Geometric local controllability: second-order conditions

Ron M. Hirschorn* Andrew D. Lewis†

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*Department of Mathematics and Statistics, Queen's University
Email: ron@mast.queensu.ca
URL: http://www.mast.queensu.ca/~ron/
†Department of Mathematics and Statistics, Queen's University
Email: andrew.lewis@queensu.ca
URL: http://www.mast.queensu.ca/~andrew/

Introduction

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- The work is based on the following premise:

  The concept of controllability is feedback invariant, so one should be able to provide feedback invariant conditions for controllability.

- During the talk, the approach we take based on this premise will be discussed, although the main results will not be stated.
- Anyone interested in seeing the details of what is presented is referred to my webpage: `http://www.mast.queensu.ca/~andrew/`

Review

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- Much has been said about the controllability of control-affine systems:

$$\dot{\xi}(t) = f_0(\xi(t)) + \sum_{a=1}^{m} u_a(t) f_a(\xi(t)).$$

- Controllability conditions typically involve brackets, and conditions that ensure that "unidirectional" brackets do not dominate their more friendly "bidirectional" friends.
- Such conditions are derived using approximations that consider the effects of brackets of various orders (nilpotent approximation).[1]
- Such conditions are rarely invariant under feedback.

[1] Contributions by various authors, including Bianchini, Hermes, Kawski, Stefani, and Sussmann.

Feedback invariance

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- Feedback equivalence: Systems $\Sigma = (M, \{f_0, f_1, \dots, f_m\})$ and $\tilde{\Sigma} = (M, \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}\})$ are related by

$$\tilde{f}_0 = f_0 + \sum_{a=1}^{m} \lambda_0^a f_a, \qquad \tilde{f}_\alpha = \sum_{a=1}^{m} \Lambda_\alpha^a f_a, \quad \alpha \in \{1, \dots, \tilde{m}\}.$$

- To ensure a feedback invariant theory, one can:
  1. fix $\{f_0, f_1, \dots, f_m\}$ and then show that one's conditions did not depend on this choice;
  2. provide a framework that is independent of any choice of $\{f_0, f_1, \dots, f_m\}$.
- We go with the latter idea.

Control-affine systems, abstractly

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- What is a control-affine system $\Sigma = (M, \{f_0, f_1, \dots, f_m\})$, really? It is an affine subbundle $\mathcal{A} \subset TM$ defined by

$$\mathcal{A}_x = \left\{ f_0(x) + \textstyle\sum_{a=1}^m u_a f_a(x) \;\middle|\; u \in \mathbb{R}^m \right\}.$$

- If the controls take their values in a set $U \subset \mathbb{R}^m$ then, for each $x \in M$, this also gives a subset $\mathscr{A}(x) \subset \mathcal{A}_x$ by

$$\mathscr{A}(x) = \left\{ f_0(x) + \textstyle\sum_{a=1}^m u_a f_a(x) \;\middle|\; u \in U \right\}.$$

- Trajectories satisfy $\dot{\xi}(t) = f_0(\xi(t)) + \displaystyle\sum_{a=1}^m u_a(t) f_a(\xi(t))$, where u is U-valued.
- Thus $\dot{\xi}(t) \in \mathscr{A}(\xi(t))$ for all t.

Affine systems

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- Generally, we shall consider an affine subbundle $\mathcal{A} \subset TM$, with no fixed set of generators in mind.
- We allow the rank of the subbundle to vary. 
- An **affine system** in $\mathcal{A}$ is then an assignment to each $x \in M$ a subset $\mathscr{A}(x) \subset \mathcal{A}_x$ (with some regularity conditions).
- A **trajectory** for an affine system is an absolutely continuous curve $\xi\colon [0, T] \to M$ having the property that $\dot{\xi}(t) \in \mathscr{A}(\xi(t))$ for a.e. $t \in [0, T]$.

Controllability of affine systems

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- Denote by $\mathcal{R}_{\mathscr{A}}(x_0, \leq T)$ the reachable set from x_0 in time at most T.
- An affine system is **small-time locally controllable** (**STLC**) from x_0 if for each $T > 0$, $x_0 \in \mathrm{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq T))$.
- We want geometric conditions $\longrightarrow$ we need controllability definitions not for $\mathscr{A}$, but for $\mathcal{A}$.

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- Call an affine system $\mathscr{A}$ **proper** at x_0 if $0_{x_0} \in \mathrm{int}(\mathrm{conv}(\mathscr{A}(x_0)))$.
- An affine subbundle $\mathcal{A}$ is **properly STLC** from x_0 if for every affine system $\mathscr{A}$ in $\mathcal{A}$,

$$\mathscr{A} \text{ proper at } x_0 \longrightarrow \mathscr{A} \text{ STLC from } x_0.$$

- An affine subbundle is **STLUC** from x_0 if for every affine system $\mathscr{A}$ in $\mathcal{A}$,

$$\mathscr{A}(x_0) \text{ compact} \longrightarrow \mathscr{A} \text{ not STLC from } x_0.$$

- Now that one has controllability *definitions* that involve only the basic geometric object $\mathcal{A}$, one would like to provide *conditions* that share this feature.

Some low-order results

First the obvious zeroth-order result. Let $L(\mathcal{A})$ be the linear part of the affine subbundle $\mathcal{A}$.

Theorem[1]

(i) If $\mathcal{A}_{x_0} = T_{x_0}M$ then $\mathcal{A}$ is properly STLC from x_0.

(ii) If $L(\mathcal{A})_{x_0} \neq \mathcal{A}_{x_0}$ then $\mathcal{A}$ is STLUC from x_0.

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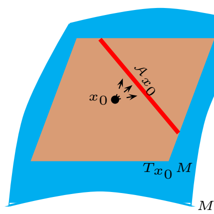
The sufficient condition is clear. Picture for the necessary condition:



[1]Sussmann, *SIAM J. Control Optim.*, **16**(5), 790–802, 1978

Now a first-order result (assume that $L(\mathcal{A})_{x_0} = \mathcal{A}_{x_0}$).

Theorem[1] *$\mathcal{A}$ is properly STLC from x_0 if*

$$\langle Z_{x_0}(\mathcal{A}), L^{(2)}(\mathcal{A})_{x_0}\rangle = T_{x_0}M.$$

Comments on our results

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- We give two "second-order" conditions, one for proper STLC and one for STLUC, for an affine subbundle $\mathcal{A}$.
- The conditions involve a vector-valued quadratic form defined on $L(\mathcal{A})_{x_0}$ and taking values in $T_{x_0}M/S_{x_0}$ for some subspace S_{x_0}.
- The nature of S_{x_0} is at the heart of the matter, and is not generally the same in both results.

[1]Bianchini/Stefani, *Internat. J. Control*, **39**(4), 701–704, 1984

Discussion

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- The development sheds some light on the relationship between controllability and feedback invariance.
- Proofs are "standard," if lengthy.
- The idea of using the affine subbundle gives results which are saying something about the "shape" of the system near the reference point $\Longrightarrow$ possibly some intuition can be acquired that can be extrapolated to higher-order conditions.