Some mechanical systems are difficult to control

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Some sample systems

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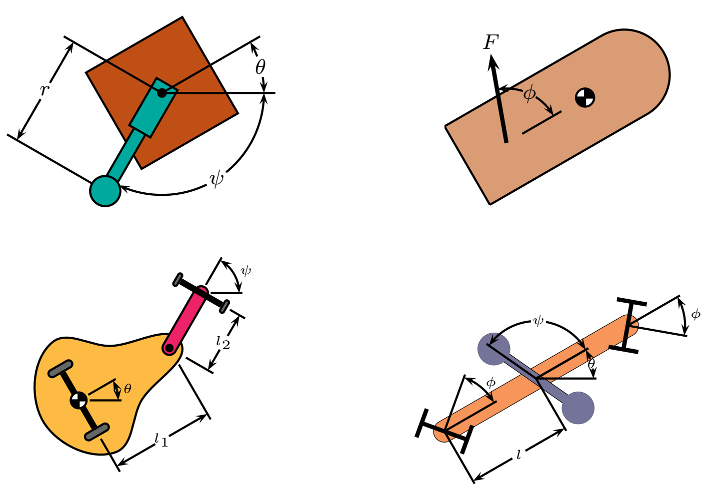

Problems we think about

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1. Describe the configurations reachable from a given point (the controllability problem).
2. Steer from point A at rest to point B at rest (the motion planning problem).
3. Design the forces, perhaps as functions of configuration, velocity, and time, so that a desired operating point is rendered stable (point stabilisation problem).
4. Follow a desired path in configuration space, possibly with a specific parameterisation (trajectory tracking problem).
5. Perform one of the above tasks in a manner that minimises some cost function (optimal control).

Something to think about while I prattle on

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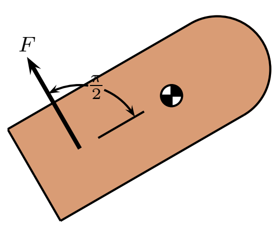
- Here's a nontrivial problem:



- Starting from rest, is it possible to reach an open set of configurations? (answer "standard")
- Starting from rest, is it possible to reach a neighbourhood of the initial configuration while not undergoing large deviations? (answer quite difficult)
- Is it possible to steer from rest in one configuration to rest in another configuration? (I do not know)

General system description

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- We consider "simple mechanical systems, possibly with constraints:"
  1. a configuration manifold Q;
  2. kinetic energy, defining a Riemannian metric $\mathbb{G}$ on Q;
  3. potential energy (a function V on Q);
  4. possibly velocity constraints that allow rolling, but not slipping (a distribution $\mathcal{D}$ on Q);
  5. a collection of forces whose direction and magnitude may be controlled (a collection $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ of vector fields).[1]
- Control system properties (consider zero potential case):
  1. inherently nonlinear;
  2. linearisations are badly behaved (linear methods not applicable);
  3. none of the "standard" nonlinear methods generally apply.

[1]Not really.

System geometry

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- The mechanical systems described above are very "structured."
- The Euler-Lagrange equations (no external forces) for the Lagrangian $L(q, \dot{q}) = \frac{1}{2}\mathbb{G}_{ij}(q)\dot{q}^i\dot{q}^j$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) - \frac{\partial L}{\partial q^i} &= \mathbb{G}_{ij}\Big[\ddot{q}^j + g^{jk}\Big(\frac{\partial \mathbb{G}_{k\ell}}{\partial q^m} - \frac{1}{2}\frac{\partial \mathbb{G}_{\ell m}}{\partial q^k}\Big)\dot{q}^\ell \dot{q}^m\Big] \\ &= \mathbb{G}_{ij}\Big[\underbrace{\ddot{q}^j + \overset{\mathrm{G}}{\Gamma}{}^j_{\ell m}\dot{q}^\ell\dot{q}^m}_{\substack{\textit{geodesic equations}\\ \text{for the } \textit{Levi-Civita}\\ \text{affine connection}}}\Big] \end{aligned}$$

$$\longrightarrow \quad \overset{\mathrm{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = 0.$$

- Even for systems with constraints, the unforced equations are geodesic.[1]

[1]Synge, *Math. Ann.*, **99**, 738–751, 1928

- This makes us think that the geometry of the affine connection may be important.
- It is in fact *extremely* important, and meshes beautifully with the control problems.
- Turns out that we may as well consider a general affine connection as modelling the unforced dynamics.

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- The effects of external forces are modelled by adding linear combinations of $\mathscr{Y}$ to the right-hand side:

$$\underbrace{\nabla_{\gamma'(t)}\nabla_{\gamma'(t)}}_{\text{acceleration}} = \underbrace{\sum_{a=1}^{m} u^a(t) Y_a(\gamma(t))}_{\frac{\text{force}}{\text{mass}}}$$

- Leads to so-called **affine connection control systems**: $\Sigma_{\mathsf{aff}} = (\mathsf{Q}, \nabla, \mathscr{Y})$.

Tools for analysis and design

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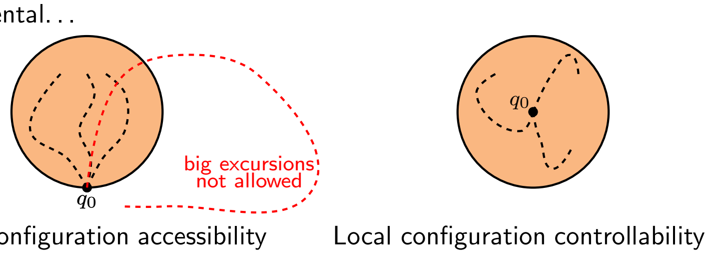
- For systems of the type we are considering, the controllability problem is fundamental...



$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u_a(t) Y_a(\gamma(t))$$

- Accessibility analysis is "standard:" Consider the following simple control system:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x).$$

- Apply the control

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$$u(t) = \begin{cases} (1,0), & 0 \leq t < \frac{T}{4} \\ (0,1), & \frac{T}{4} \leq t < \frac{T}{2} \\ (-1,0), & \frac{T}{2} \leq t < \frac{3T}{4} \\ (0,-1), & \frac{3T}{4} \leq t \leq T. \end{cases}$$

- Where does $x(T)$ end up?

- We determine that

$$x(T) = x(0) + \sqrt{T}\,[f_1, f_2](x(0)) + \text{h.o.t.}, \qquad [f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2.$$

- $[f_1, f_2]$ is the **Lie bracket** of f_1 and f_2.
- More generally, we may consider a control system like

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$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u_a(t) f_a(x(t)).$$

- By applying suitable controls, one may move in the directions

$$\begin{aligned} & f_0,\ f_1, \dots, f_m, \\ & [f_a, f_b], \qquad a, b = 0, \dots, m, \\ & [f_a, [f_b, f_c]], \qquad a, b, c = 0, \dots, m, \\ & \text{etc.} \end{aligned}$$

A simple exhibition of the Lie bracket

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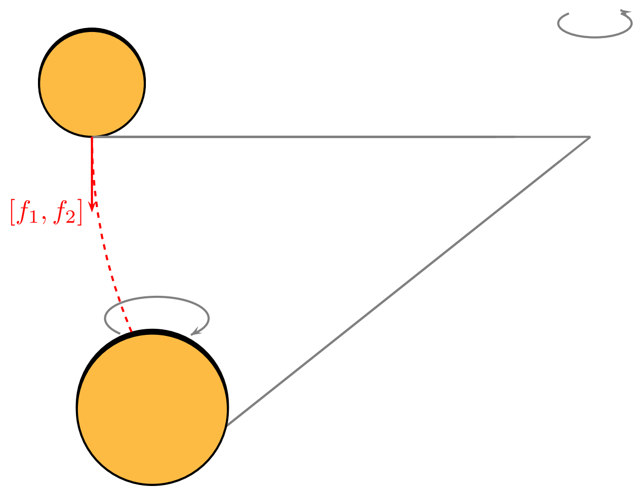

Some controllability results for mechanical systems

Accessibility

- For mechanical systems, the interaction of the Lie bracket and the system geometry (i.e., the affine connection) is very attractive. This gives nice accessibility results.[1][2][3]
- Associated with ∇ define the **symmetric product**:

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$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

- Let Y be the distribution generated by $\mathscr{Y}$.
- Let $\mathrm{Sym}^{(\infty)}(\mathsf{Y})$ be the distribution generated by $\mathscr{Y}$ under symmetric product.
- Let $\mathrm{Lie}^{(\infty)}(\mathscr{V})$ be the distribution generated by a family of vector fields $\mathscr{V}$ under Lie bracket.

[1] L/Murray, *SIAM Review*, **41**(3), 555–574, 1999
[2] L/Murray *Systems Control Lett.*, **31**(4), 199–205, 1997
[3] L, *Rep. Math. Phys.*, **42**(1/2), 135–164, 1998

- For $q_0 \in \mathsf{Q}$ and $T > 0$ let $\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T) \subset \mathsf{TQ}$ be the set of states reachable from 0_{q_0} in time at most T and let $\mathcal{R}_{\mathsf{Q}}(q_0, \leq T) = \tau_{\mathsf{Q}}(\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T))$.

Definition 1 $\Sigma_{\text{aff}} = (\mathsf{Q}, \nabla, \mathscr{Y})$ is

(i) ***accessible*** from 0_{q_0} if $\operatorname{int}(\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T)) \neq \varnothing$ for all sufficiently small T and is

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(ii) ***configuration accessible*** from q_0 if $\operatorname{int}(\mathcal{R}_{\mathsf{Q}}(q_0, \leq T)) \neq \varnothing$ for all sufficiently small T.

Theorem 1 $\Sigma_{\text{aff}} = (\mathsf{Q}, \nabla, \mathscr{Y})$ *is*

(i) accessible from 0_{q_0} *if and only if* $\mathrm{Sym}^{(\infty)}(\mathcal{Y})_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$ *and is*

(ii) configuration accessible from q_0 *if and only if* $\mathrm{Lie}^{(\infty)}(\mathrm{Sym}^{(\infty)}(\mathcal{Y}))_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$.

Controllability

Definition 2 $\Sigma_{\text{aff}} = (\mathsf{Q}, \nabla, \mathscr{Y})$ is

(i) ***small-time locally controllable*** (***STLC***) from 0_{q_0} if $0_{q_0} \in \operatorname{int}(\mathcal{R}_{\mathsf{TQ}}(q_0, \leq T))$ for all sufficiently small T, and is

(ii) ***small-time locally configuration controllable*** (***STLCC***) from q_0 if $q_0 \in \operatorname{int}(\mathcal{R}_{\mathsf{Q}}(q_0, \leq T))$ for all sufficiently small T.

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- L/Murray give sufficient conditions involving symmetric products and based on work of Sussmann.[1]
- These sufficient conditions lead to a class of control algorithms for certain systems that rely on specially constructed periodic inputs.[2]
- Problems treated include the steering problem, the point stabilisation problem, and the trajectory tracking problem.
- *Movies*

[1] *SIAM J. Control Optim.*, **25**(1), 158–194, 1987

[2] Bullo/Leonard/L, *IEEE Trans. Automat. Control*, **45**(8), 1437–1454, 2000

Low-order controllability results[1,2]

- These revolve around vector-valued quadratic forms.
- For $\mathbb{R}$-vector spaces V and W, let $\mathsf{TS}^2(V;W)$ be the collection of symmetric bilinear maps $B\colon V \times V \to W$.
- For $B \in \mathsf{TS}^2(V;W)$ and $\lambda \in W^*$ define $\lambda B(w_1, w_2) = \langle \lambda; B(w_1, w_2)\rangle \in \mathbb{R}$.

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Definition 3 $B \in \mathrm{TS}^2(V;W)$ is

(i) ***indefinite*** if for each $\lambda \in W^*$, λB is neither positive nor negative-semidefinite and is

(ii) ***definite*** if there exists $\lambda \in W^*$ so that λB is positive-definite.

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[1] Hirschorn/L, *Proceedings of 40th IEEE CDC*, 4216-4221, Dec. 2001.
[2] Bullo/L, Submitted to *SIAM J. Control Optim.*, Jan. 2003.

- For $q \in \mathsf{Q}$ define $B_{\mathcal{Y}}(q) \in \mathsf{TS}^2(\mathcal{Y}_q; \mathsf{T}_q\mathsf{Q}/\mathcal{Y}_q)$ by

$$B_{\mathcal{Y}}(q)(v_1, v_2) = \pi_{\mathcal{Y}_q}(\langle V_1 : V_2\rangle(q)),$$

where V_1 and V_2 are vector fields extending $v_1, v_2 \in \mathcal{Y}_q$.

Theorem 2 *Let $\Sigma_{\mathrm{aff}} = (\mathsf{Q}, \nabla, \mathscr{Y})$. If $q_0 \in \mathsf{Q}$ is a regular point of $\mathcal{Y}$ then Σ_{aff} is*

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(i) not STLCC from q_0 if $B_{\mathcal{Y}}(q_0)$ is definite.

Assume that $\mathrm{Sym}^{(\infty)}(\mathcal{Y})_{q_0}$ is generated by symmetric products of degree at most two. Then Σ_{aff} is

(ii) STLC from 0_{q_0} if $\mathrm{Sym}^{(\infty)}(\mathcal{Y})_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$ and if $B_{\mathcal{Y}}(q_0)$ is indefinite, and is

(iii) STLCC from q_0 if $\mathrm{Lie}^{(\infty)}(\mathrm{Sym}^{(\infty)}(\mathcal{Y}))_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$ and if $B_{\mathcal{Y}}(q_0)$ is indefinite.

Application to motion planning

- A vector field X is a **decoupling vector field** for Σ_{aff} if one can follow any reparameterisation of any integral curve of X with a controlled trajectory of Σ_{aff}.
- Think of $B_{\mathcal{Y}}$ as a bundle mapping on all of Q.

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Theorem 3[1] *X is a decoupling vector field if and only if $X \in \Gamma^\infty(\mathcal{Y})$ and $B_{\mathcal{Y}}(X, X) = 0$.*

- If there exists generators $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$ for $\mathcal{Y}$ that are all decoupling vector fields, then $B_{\mathcal{Y}}(q)$ is indefinite for each $q \in$ Q. (If $\mathrm{codim}(\mathcal{Y}) = 1$ then the converse is also true.)
- For each of the example systems that is STLCC, $\mathcal{Y}$ possesses a collection of decoupling generators!
- *Movies*

[1] Bullo/Lynch, *IEEE Trans. Robotics and Autom.*, **17**(4), 402–412, 2001.