From controllability to motion planning for mechanical systems

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Some sample systems

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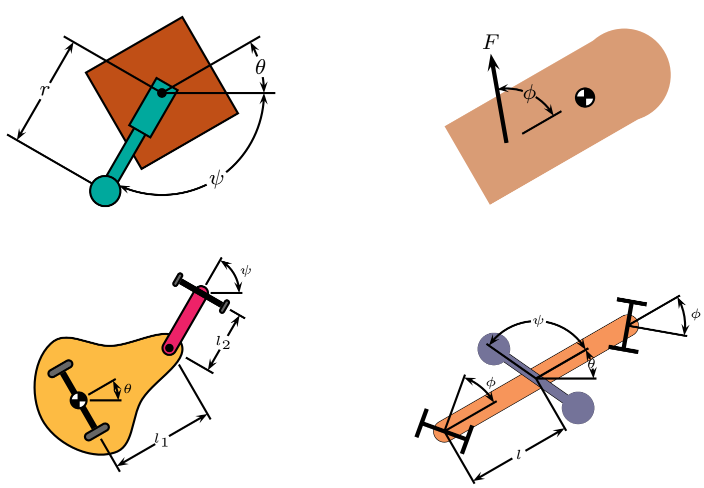

What are we interested in?

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- Broadly, a general methodology that encompasses modelling, analysis, and design.
- More specifically, for one of the example systems, or any system "like" them,
  - can we model it in a unified manner that is conducive to the further objectives of analysis and design?
  - can one describe its reachable set?
  - if given a suitable cost function, can one analyse the corresponding extremals of the optimal control problem?
  - are there simple collections of trajectories that are sufficiently rich to do motion planning?

Modelling

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- For us, a **simple mechanical control system** consists of a 6-tuple $(\mathsf{Q}, \mathbb{G}, V, F, \mathcal{D}, \mathscr{F} = \{F^1, \dots, F^m\})$ where
  1. Q is a finite-dimensional configuration manifold,
  2. $\mathbb{G}$ is a Riemannian metric on Q,
  3. V is a potential function on Q,
  4. F represents all non-potential forces that are not controlled (e.g., dissipative forces),
  5. $\mathcal{D}$ is a distribution on Q modelling linear velocity constraints,
  6. $\mathscr{F}$ is a collection of one-forms on Q, each representing a force over which we have control.
- The equations of motion are the Euler-Lagrange equations with Lagrangian $L(v_q) = \frac{1}{2}\mathbb{G}(v_q, v_q) - V(q)$, with external force $F + \sum_{a=1}^m u_a F^a$, and subject to the nonholonomic constraints specified by $\mathcal{D}$.

- We generally simplify to the situation where $V = 0$ and $F = 0$, although potential forces have received some attention,[1] as have dissipative forces.[2]
- With these simplifications, the problem is reduced to an **affine connection control system** which is described by a 4-tuple $\Sigma_{\mathsf{aff}} = (\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y} = \{Y_1, \dots, Y_m\})$ with

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1. Q as before,
2. ∇ an affine connection (which is not generally Levi-Civita),
3. $\mathcal{D}$ a distribution to which ∇ restricts,
4. $\mathscr{Y}$ a collection of vector fields on Q (these are related to the one-forms $\mathscr{F}$).

[1] L/Murray, *SIAM J. Control Optim.*, **35**(3), 766–790, 1997.
[2] Cortés/Martínez/Bullo, *IEEE Trans. Automat. Control*, submitted, July 2001.

- When $\mathcal{D} = \mathsf{TQ}$ then ∇ is the Levi-Civita affine connection $\overset{\mathbb{G}}{\nabla}$ associated with $\mathbb{G}$.
- When $\mathcal{D} \subsetneq \mathsf{TQ}$ then ∇ is defined by

$$\nabla_X Y = \overset{\mathbb{G}}{\nabla}_X Y - (\overset{\mathbb{G}}{\nabla}_X P)(Y),$$

where P is the orthogonal projection onto $\mathcal{D}^\perp$.

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- The equations of motion for such systems are

$$\boxed{\nabla_{\dot\gamma(t)}\dot\gamma(t) = \sum_{a=1}^{m} u_a(t) Y_a(\gamma(t))}$$

for a controlled trajectory (γ, u) satisfying $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for some (and hence all) t.

Controllability

- Questions: Starting from rest at $q_0 \in \mathsf{Q}$ does the set of reachable configurations
  1. have a nonempty interior? (accessibility)
  2. contain q_0 in its interior? (controllability)

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- Accessibility is "easy" and beautiful (combine Sussmann/Jurdjevic with affine differential geometry)[1].
- Controllability is quite difficult. Preliminary (and quite unsatisfactory) results were found by L/Murray.[2]
- It is possible to show that any (analytic) single-input system will be controllable only on a strict analytic subset.

[1] L/Murray, *SIAM Review*, **41**(3), 555–574, 1999
[2] Ibid.

- Let's consider an example:

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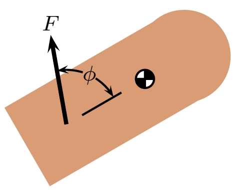
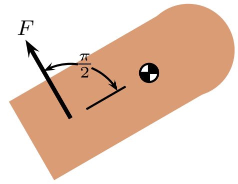
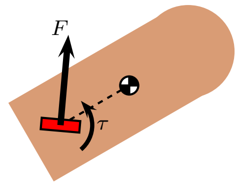


- accessible
- controllable ("easy")



- accessible
- not controllable (not so "easy")

- Add more stuff to the model:



- Controllability now goes from "not so easy" to "requiring new techniques."[1,2]

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- The new techniques involve the vector-valued quadratic form

$$\begin{aligned} B_{\mathcal{Y}}(q_0)\colon \mathcal{Y}_{q_0} \times \mathcal{Y}_{q_0} &\to \mathsf{T}_{q_0}\mathsf{Q}/\mathcal{Y}_{q_0} \\ (v_1, v_2) &\mapsto \pi_{\mathcal{Y}_q}(\langle V_1 : V_2\rangle(q_0)), \end{aligned}$$

  where

$$\langle V_1 : V_2\rangle = \nabla_{V_1}V_2 + \nabla_{V_2}V_1$$

  is the **symmetric product**.

[1]Hirschorn/L, *Proceedings of 40th IEEE CDC*, 4216–4221, Dec. 2001.
[2]Bullo/L, submitted to *SIAM J. Control Optim.*, January 2003.

- Using the vector-valued quadratic form ideas one can prove a general result for two-input affine connection control systems which says, roughly, that they are either controllable in a very nice way, or they are controllable only on an analytic set.[1]

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- The hovercraft with the fan dynamics is of the "only controllable on an analytic set" sort.
- Is there some sort of measure of "robustness" of controllability?

[1]Tyner/L, submitted to CDC03.

Motion planning

- Question: If a system is controllable, is it possible to steer from rest at $q_1 \in \mathsf{Q}$ to rest at $q_2 \in \mathsf{Q}$?
- The approach is to find a collection of "motion primitives" that are rich enough to allow one to solve the motion planning problem by concatenation of primitives.

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- What sort of primitives should one look for?
- We consider **decoupling vector fields**. These are vector fields on Q whose integral curves, and any reparameterisation of them, can be followed by trajectories of the mechanical system.
- The idea is that given a rich enough class of decoupling vector fields, one solves the motion planning problem by concatenating their integral curves.

- There is a nontrivial connection between the vector-valued quadratic form used in controllability and the notion of a decoupling vector field:

  Theorem[1] *X is a decoupling vector field if and only if X is $\mathcal{Y}$-valued and $B_{\mathcal{Y}}(X, X) = 0$.*

  "Theorem" *If $\dim(\mathcal{Y}) = \dim(\mathsf{Q}) - 1$ then the existence of enough decoupling vector fields for motion planning can be decided using $B_{\mathcal{Y}}$.*

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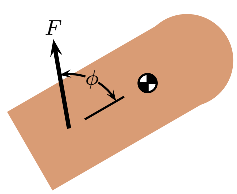
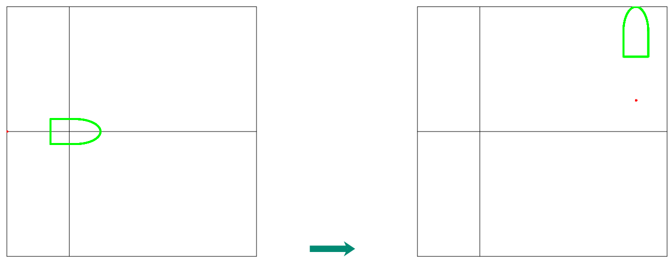
- What about our example?



  - Controllable, as we have seen.
  - Possible to find enough decoupling vector fields.
  - There are two. What are they?

[1] Bullo/Lynch, *IEEE Trans. Robotics and Autom.*, **17**(4), 402–412, 2001.



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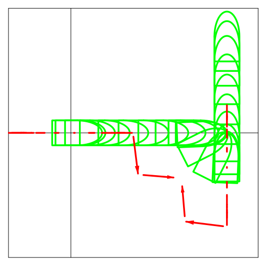
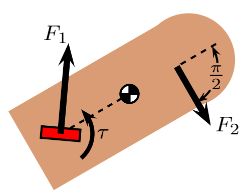


- What about the more complicated model?
- It is not controllable, so it needs one more input:



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- The theory predicts there are enough decoupling vector fields to do motion planning.
- Last week Dave Tyner found them.
- Are they simple enough to do anything with?

A not so easy example

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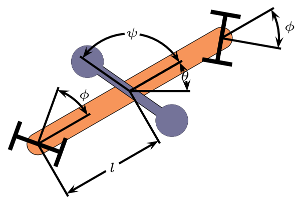
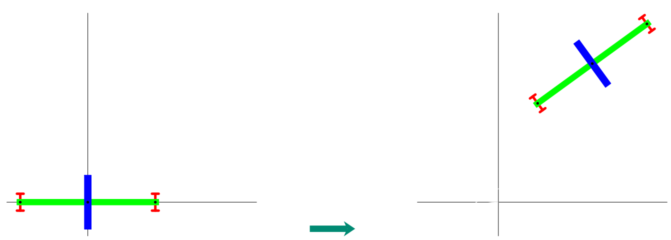


- accessible.
- controllable.

- The system also possess enough decoupling vector fields to do motion planning.
- This can be done explicitly!



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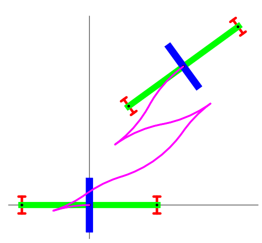

What else?

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- We have an actual hovercraft, and the open-loop motion planning primitives work *extremely* poorly.
- Linearise around trajectories to stabilise them in closed-loop.
- Understand non-ideal model effects (friction, actuator magnitude and rate constraints, etc.)