The role of controllability in motion planning for affine connection control systems

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Affine connection control systems

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- An **affine connection control system** is a 4-tuple $(\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y})$ where
  1. Q is the configuration manifold,
  2. ∇ is an affine connection on Q,
  3. $\mathcal{D}$ is a constant rank distribution on Q that in invariant under ∇, and
  4. $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ are $\mathcal{D}$-valued vector fields on Q.
- The control equations are

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t)).$$

- These equations model mechanical systems with a kinetic energy Lagrangian, nonholonomic constraints modelled by $\mathcal{D}$, and an external force that is a user-specified linear combination of the vector fields in $\mathscr{Y}$.

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- For unconstrained systems take
  1. $\mathcal{D} = \mathsf{T}Q$,
  2. $\nabla = \overset{\mathbb{G}}{\nabla}$, the Levi-Civita connection for the kinetic energy metric $\mathbb{G}$, and
  3. $Y_a = \mathbb{G}^\sharp(F^a)$, $a \in \{1, \dots, m\}$, where $F^1, \dots, F^m$ are the physical forces.
- For constrained systems take
  1. $\mathcal{D} \subsetneq \mathsf{T}Q$,
  2. ∇ is the affine connection

     $$\nabla_X Y = \overset{\mathbb{G}}{\nabla}_X Y - (\overset{\mathbb{G}}{\nabla}_X P^{\mathcal{D}^\perp})(Y),$$

     where $P^{\mathcal{D}^\perp}$ is the orthogonal projection onto $\mathcal{D}^\perp$, and
  3. $Y_a = P^{\mathcal{D}}(\mathbb{G}^\sharp(F^a))$, $a \in \{1, \dots, m\}$, where $P^{\mathcal{D}}$ is the orthogonal projection onto $\mathcal{D}$.

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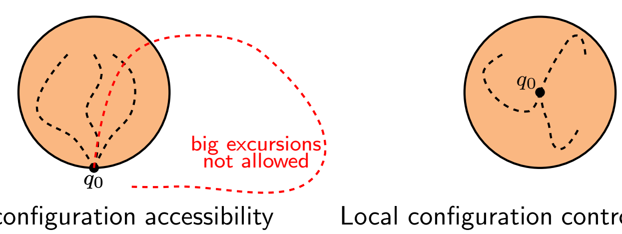
Controllability

- For systems and problems of the type we are considering, the controllability problem is fundamental...



Local configuration accessibility   Local configuration controllability

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u_a(t) Y_a(\gamma(t))$$

Accessibility

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- For studies of controllability for general nonlinear systems of the form
$$\dot{\xi}(t) = f_0(\xi(t)) + \sum_{a=1}^{m} u^a(t) f_a(\xi(t)),$$
the Lie brackets of the vector fields $\{f_0, f_1, \ldots, f_m\}$ play a fundamental rôle.
- For mechanical systems, the interaction of the Lie bracket and the system geometry (i.e., the affine connection) is very attractive. This gives nice accessibility results.[1,2,3]
- An important rôle in these results is played by the **symmetric product** associated with ∇:
$$\langle X : Y\rangle = \nabla_X Y + \nabla_Y X.$$

[1] L/Murray, *SIAM Review*, **41**(3), 555–574, 1999
[2] L/Murray *Systems Control Lett.*, **31**(4), 199–205, 1997
[3] L, *Rep. Math. Phys.*, **42**(1/2), 135–164, 1998

Controllability

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- L/Murray give sufficient conditions involving symmetric products and based on work of Sussmann.[1]
- These sufficient conditions lead to a class of control algorithms for certain systems that rely on specially constructed periodic inputs.[2]
- Problems treated include the steering problem, the point stabilisation problem, and the trajectory tracking problem.
- The conditions of L/Murray are not entirely satisfactory since there are systems that fail the conditions but are (trivially) controllable.

[1] *SIAM J. Control Optim.*, **25**(1), 158–194, 1987
[2] Bullo/Leonard/L, *IEEE Trans. Automat. Control*, **45**(8), 1437–1454, 2000

Low-order controllability[1,2]

- These revolve around vector-valued quadratic forms.
- For $\mathbb{R}$-vector spaces V and W, let $\mathsf{TS}^2(V;W)$ be the collection of symmetric bilinear maps $B\colon V\times V\to W$.
- For $B\in\mathsf{TS}^2(V;W)$ and $\lambda\in W^*$ define $\lambda B(w_1,w_2)=\langle\lambda;B(w_1,w_2)\rangle\in\mathbb{R}$.

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Definition 1 $B\in\mathrm{TS}^2(V;W)$ is

(i) ***indefinite*** if for each $\lambda\in W^*$, λB is neither positive nor negative-semidefinite and is

(ii) ***definite*** if there exists $\lambda\in W^*$ so that λB is positive-definite.

[1] Hirschorn/L, *Proceedings of 40th IEEE CDC*, 4216-4221, Dec. 2001.
[2] Bullo/L, Submitted to *SIAM J. Control Optim.*, Jan. 2003.

- For $q\in\mathsf{Q}$ define $B_{\mathcal{Y}}(q)\in\mathsf{TS}^2(\mathcal{Y}_q;\mathsf{T}_q\mathsf{Q}/\mathcal{Y}_q)$ by

$$B_{\mathcal{Y}}(q)(v_1,v_2)=\pi_{\mathcal{Y}_q}(\langle V_1:V_2\rangle(q)),$$

where V_1 and V_2 are vector fields extending $v_1,v_2\in\mathcal{Y}_q$.

Theorem 1 *Let $\Sigma_{\mathrm{aff}}=(\mathsf{Q},\nabla,\mathscr{Y})$. If $q_0\in\mathsf{Q}$ is a regular point of $\mathcal{Y}$ then Σ_{aff} is*

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(i) not configuration controllable from q_0 if $B_{\mathcal{Y}}(q_0)$ is definite.

Assume that $\mathrm{Sym}^{(\infty)}(\mathcal{Y})_{q_0}$ is generated by symmetric products of degree at most two. Then Σ_{aff} is

(ii) controllable from 0_{q_0} if $\mathrm{Sym}^{(\infty)}(\mathcal{Y})_{q_0}=\mathsf{T}_{q_0}\mathsf{Q}$ and if $B_{\mathcal{Y}}(q_0)$ is indefinite, and is

(iii) configuration controllable from q_0 if $\mathrm{Lie}^{(\infty)}(\mathrm{Sym}^{(\infty)}(\mathcal{Y}))_{q_0}=\mathsf{T}_{q_0}\mathsf{Q}$ and if $B_{\mathcal{Y}}(q_0)$ is indefinite.

An example

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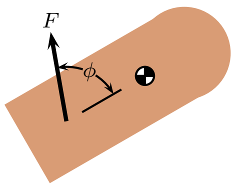
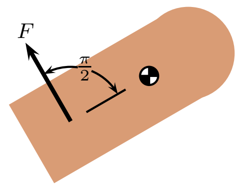
- Let's consider an example:





  - accessible
  - controllable ("easy")

  - accessible
  - not controllable (not so "easy")

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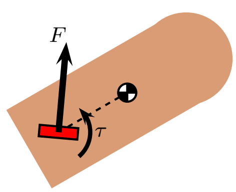
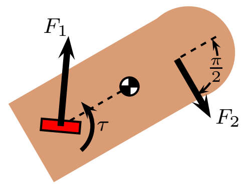
- Add more stuff to the model:



- Controllability now goes from "not so easy" to "requiring new techniques."
- One can prove a general result concerning two-input systems which states that a large class of two-input systems are either controllable in a nice way or only controllable on an analytic set.
- The model above is of the latter sort.

- Add another input:



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- The quadratic form low-order controllability results may be used to predict that the system is controllable.
- This really appears to need the sophisticated quadratic form methods.

So your system is controllable?

Kinematic controllability

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- It turns out that for many systems, satisfaction of the low-order controllability results leads to simple motion planning strategies.
- These are based on the notion of a decoupling vector field,[1] which is a vector field all of whose integral curves can be followed with an arbitrary reparameterisation.
- The idea is that given a rich enough class of decoupling vector fields, one solves the motion planning problem by concatenating their integral curves.

[1] Bullo/Lynch, *IEEE Trans. Robotics and Autom.*, **17**(4), 402–412, 2001.

- There is a nontrivial connection between the vector-valued quadratic form used in controllability and the notion of a decoupling vector field:

  Theorem [1] *X is a decoupling vector field if and only if X is $\mathcal{Y}$-valued and $B_{\mathcal{Y}}(X,X)=0$.*

  Theorem [2] *If there exists generators $\mathscr{Y}=\{Y_1,\dots,Y_m\}$ for $\mathcal{Y}$ that are all decoupling vector fields, then $B_{\mathcal{Y}}(q)$ is indefinite for each $q\in\mathsf{Q}$. (If $\operatorname{codim}(\mathcal{Y})=1$ then the converse is also true.)*

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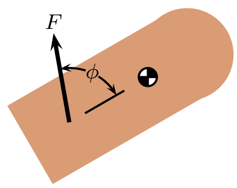
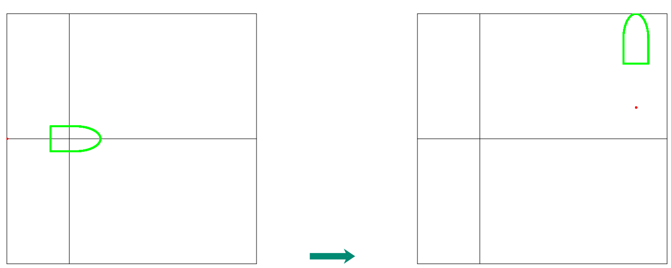
Planar body example



- Controllable, as we have seen.
- Possible to find enough decoupling vector fields.
- There are two. What are they?

[1] Bullo/Lynch, *IEEE Trans. Robotics and Autom.*, **17**(4), 402–412, 2001.
[2] Bullo/L, Submitted to *SIAM J. Control Optim.*, Jan. 2003.



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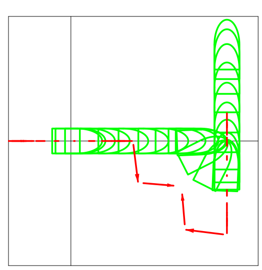

More complicated planar body example

- What about the more complicated model?
- It is not controllable, so it needs one more input:

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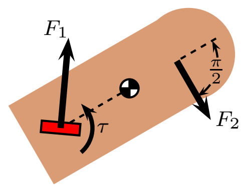
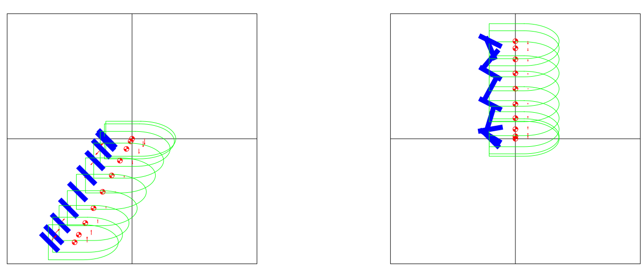


- The theory predicts there are enough decoupling vector fields to do motion planning.
- A month ago Dave Tyner found them.



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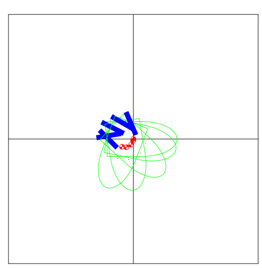

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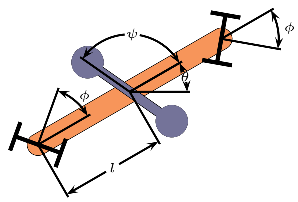
A constrained example



- accessible.
- controllable.

- The system also possess enough decoupling vector fields to do motion planning.
- This can be done explicitly!

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