Affine connection control systems with interesting controllability properties

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Some illustrative "real" examples

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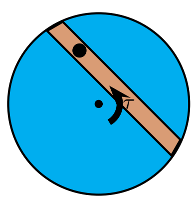
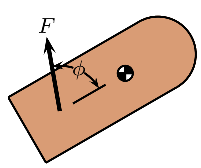
- Shen/Sanyal/McClamroch, MTNS02.

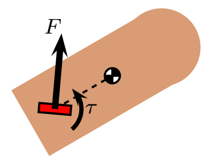


- Controllable only from configurations where ball is at centre of disk.
- Simple hovercraft model.



- Controllable from all configurations.

- Slightly more realistic hovercraft model.



- Only controllable from these configurations:

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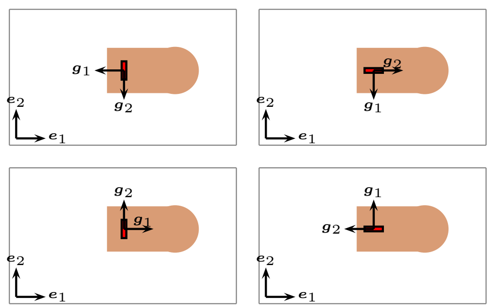

A too quick overview of controllability

- The methods we use for answering this question are based on low-order controllability results of Hirschorn/L, Bullo/L, and Tyner/L.

The setup

- **Affine connection control system**: $(\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y})$ where
  1. Q is the configuration manifold,
  2. $\mathcal{D}$ is a regular distribution on Q,
  3. ∇ is an affine connection on Q which restricts to ∇, and
  4. $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ are vector fields taking values in $\mathcal{D}$.

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- All data analytic.
- Governing equations:

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t)).$$

- This can be used to model mechanical systems described by
  1. a kinetic energy Lagrangian,
  2. (possibly) nonholonomic constraints,
  3. input forces whose directions vary only with position,
  4. no non-control external forces.

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- *The question*: If Q_c denotes the configurations from which the system is controllable, and Q_u denotes the configurations from which the system is uncontrollable, what can these sets look like?
- For example, for the two example systems, Q_c is a strict analytic subset of Q,[1] i.e., Q_c is extremely small.

[1] An ***analytic subset*** is one that is locally the intersection of the zeros of a finite number of analytic functions.

Vector-valued quadratic forms

- Let $\mathsf{TS}^2(\mathsf{V};\mathsf{U})$ be the set of symmetric bilinear maps from $\mathsf{V}\times\mathsf{V}$ to U.
- For $\lambda\in\mathsf{U}^*$ and $B\in\mathsf{TS}^2(\mathsf{V};\mathsf{U})$ let $\lambda B\in\mathsf{TS}^2(\mathsf{V};\mathbb{R})$ be defined by $\lambda B(v_1,v_2)=\langle\lambda;B(v_1,v_2)\rangle$.

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- $B\in\mathsf{TS}^2(\mathsf{V};\mathsf{U})$ is
  1. **definite** if there exists $\lambda\in\mathsf{U}^*$ so that λB is positive-definite, is
  2. **semidefinite** if there exists $\lambda\in\mathsf{U}^*$ so that λB is positive-semidefinite (but nonzero), and is
  3. **indefinite** if for each $\lambda\in\mathsf{U}^*$, λB is not semidefinite.

A vector-valued quadratic form for controllability

- $(\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y})$ an affine connection control system.
- Let $\mathcal{Y}$ be the distribution generated by the vector fields $\mathscr{Y}$.
- Define the **symmetric product**: $\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$.
- For $q \in \mathsf{Q}$ define $B_{\mathcal{Y}_q} \in \mathsf{TS}^2(\mathcal{Y}_q; \mathsf{T}_q\mathsf{Q}/\mathcal{Y}_q)$ by

$$B_{\mathcal{Y}_q}(u, v) = \pi_{\mathcal{Y}_q}(\langle U : V \rangle(q)),$$

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where U and V are vector fields extending $u, v \in \mathcal{Y}_q$, and $\pi_{\mathcal{Y}_q} \colon \mathsf{T}_q\mathsf{Q} \to \mathsf{T}_q\mathsf{Q}/\mathcal{Y}_q$ is the canonical projection.

- *Intuition*: $B_{\mathcal{Y}_q}$ encodes information about the "lowest-order good and bad symmetric products."[1]
- $B_{\mathcal{Y}_q}$ indefinite $\Longleftrightarrow$ "good" $\Longleftrightarrow$ "controllable"
- $B_{\mathcal{Y}_q}$ definite $\Longleftrightarrow$ "bad" $\Longleftrightarrow$ "uncontrollable"

[1]As per Sussmann, *SIAM J. Control Optim.*, **25**(1), 158–194, 1987.

- For precise statement of quadratic form controllability results, see papers by Bullo/Hirschorn/L.[1]

Relation to motion planning primitives

- A vector field X on Q is a **decoupling vector field** for an affine connection control system $\Sigma = (\mathsf{Q}, \nabla, \mathcal{D}, \mathscr{Y})$ if every reparameterisation of every integral curve of X can be followed by a trajectory of Σ.[2]
- *Intuition*: Given enough decoupling vector fields, one can steer the system between configurations by following suitable concatenations of decoupling vector fields.

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- $\Longrightarrow$ decoupling vector fields are good (actually, they are "good").
- **Fact:** X is decoupling if and only if X is $\mathcal{Y}$-valued and $B_{\mathcal{Y}}(X, X) = 0$.
- $\Longrightarrow$ vector-valued quadratic form gives relationship between controllability and motion planning algorithms.

[1]Hirschorn/L, *CDC Proceedings*, 4216–4221, 2001, and Bullo/L, submitted to *SIAM J. Control Optim.*, Jan. 2003.
[2]Bullo/Lynch, *IEEE Trans. Robotics and Autom.*, **17**(4), 402–412, 2001.

Some answers to "The question"

The single-input case ($\dim(\mathsf{Q}) > 1$)

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- For $\lambda \in \Gamma(\mathrm{ann}(\mathcal{Y}))$ define a function $f_\lambda(q) = \langle \lambda(q); \langle Y_1 : Y_1 \rangle(q) \rangle$.
- The controllability results say that the system is controllable at q only if $f_\lambda(q) = 0$ for every $\lambda \in \Gamma(\mathrm{ann}(\mathcal{Y}))$.
- System is analytic $\longrightarrow$ the system is controllable from at most a strict analytic subset of Q.
- This covers the Shen/Sanyal/McClamroch example.

The $(n-1)$-input case

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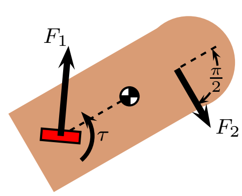
- Strong relationship between controllability and motion planning.
- **Fact:** If $B_\mathcal{Y}$ is indefinite at every point, then there are $n-1$ decoupling vector fields. *This is affine connection control system heaven.*
- Back to hovercraft model, and add an input:



- Now $\dim(\mathsf{Q}) = 4$ and $\mathrm{rank}(\mathcal{Y}) = 3$.
- One can compute directly that $B_\mathcal{Y}$ is indefinite at every point in Q $\longrightarrow$ there are three decoupling vector fields.
- What are they? Dave Tyner found them by fiddling with quadratic forms!

The two-input case ($\dim(\mathsf{Q}) > 2$)

- Let us categorise by number of decoupling vector fields.

1. No decoupling vector fields.
   (a) System can be controllable everywhere ($\mathsf{Q}_c = \mathsf{Q}$). Take

$$\begin{aligned}\ddot{x}_1 &= u_1\\ \ddot{x}_2 &= u_2\\ \ddot{x}_3 &= -\dot{x}_1^2 + \dot{x}_2^2\\ \ddot{x}_4 &= 2\dot{x}_1\dot{x}_2.\end{aligned}$$

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   (*Open problem*: How to do motion planning for this system.)

   (b) System can be controllable nowhere ($\mathsf{Q}_u = \mathsf{Q}$). Take

$$\begin{aligned}\ddot{x}_1 &= u_1\\ \ddot{x}_2 &= u_2\\ \ddot{x}_3 &= \dot{x}_1^2 + \dot{x}_2^2.\end{aligned}$$

   (c) Q_c can be a strict analytic subset. Take

$$\begin{aligned}\ddot{x}_1 &= u_1\\ \ddot{x}_2 &= u_2\\ \ddot{x}_3 &= \dot{x}_1^2 + x_2\dot{x}_2^2.\end{aligned}$$

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   (d) Q_c and Q_u can both have nonempty interior. Take

$$\begin{aligned}\ddot{x}_1 &= u_1\\ \ddot{x}_2 &= u_2\\ \ddot{x}_3 &= 2\dot{x}_1\dot{x}_2\\ \ddot{x}_4 &= x_1\dot{x}_1^2 + x_2\dot{x}_2^2.\end{aligned}$$

2. One decoupling vector field.

   (a) Q_c is a strict analytic subset of Q[1] (this covers the 4DOF hovercraft with two inputs).

3. Two decoupling vector fields.

   (a) Controllable (more generally, basis of decoupling vector fields ⟹ controllable).

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- The "transition" cases where $B_{\mathcal{Y}}$ is semidefinite have to be considered "by hand."
- A higher-order theory would cover these.
- An interesting observation: in all cases, Q_c is closed.
- Is this always the case?

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[1] This is a result of Tyner/L.