Jacobian linearisation in a geometric setting

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First of all, has anybody done this?

Where this originated

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- We are trying to do motion planning for a difficult to control hovercraft.

- We have open-loop trajectory generation schemes that perform very poorly.
- We need feedback; linearisation about open-loop trajectories is controllable $\Longrightarrow$ Use standard linearisation techniques.

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- The hovercraft system is a mechanical system with special structure.
- Its linearisation has associated with it some nice geometry (related to the Jacobi equation of geodesic variation).
- The nice geometry appears to be less well-developed for general control-affine systems.

Some questions concerning the usual technique

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- On $\mathcal{U} \Subset \mathbb{R}^n$ consider the control-affine system

$$\dot{\boldsymbol{\gamma}}(t) = \boldsymbol{f}_0(\boldsymbol{\gamma}(t)) + \sum_{a=1}^{m} u^a(t)\boldsymbol{f}_a(\boldsymbol{\gamma}(t))$$

with reference trajectory $(\boldsymbol{\gamma}_{\mathsf{ref}}, \boldsymbol{u}_{\mathsf{ref}})$.

- Linearise in the usual manner:

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{A}(t)\boldsymbol{\xi}(t) + \boldsymbol{B}\boldsymbol{v}(t),$$

where

$$\boldsymbol{A}(t) = \boldsymbol{D}\boldsymbol{f}_0(\boldsymbol{\gamma}_{\mathsf{ref}}(t)) + \sum_{a=1}^{m} u^a_{\mathsf{ref}}(t)\boldsymbol{f}_a(\boldsymbol{\gamma}_{\mathsf{ref}}(t))$$

$$\boldsymbol{B}(t) = \left[\ \boldsymbol{f}_1(\boldsymbol{\gamma}_{\mathsf{ref}}(t)) \ \middle|\ \cdots \ \middle|\ \boldsymbol{f}_m(\boldsymbol{\gamma}_{\mathsf{ref}}(t))\ \right].$$

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- Now consider a control-affine system on a manifold M:

$$\gamma'(t) = f_0(\gamma(t)) + \sum_{a=1}^{m} u^a(t) f_a(\gamma(t))$$

with reference trajectory $(\gamma_{\mathsf{ref}}, u_{\mathsf{ref}})$.

- How do you linearise this *in a coordinate-independent manner*?
(There *is* a problem here since the families of linear maps $\{\boldsymbol{A}(t)\}$ and $\{\boldsymbol{B}(t)\}$ defined above are *not* coordinate-independent.)

- Some questions:
  1. What replaces the Jacobian?
  2. Where does the linearisation live? It does not live on a vector space (at least not a finite-dimensional one), as in the usual case.
  3. How do you check, or even define, the controllability of the linearisation? The controllability Gramian no longer makes sense.
  4. How is the stability of the linearisation defined?
  5. How is the linearisation stabilised by linear feedback?
  6. If one stabilises the linearisation, how can the resulting linear feedback be implemented on the nonlinear system?
  7. If the closed-loop system is suitably stable, does this imply closed-loop stability for the nonlinear system?

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Formulation of problem

- The objective is to strip away all unnecessary structure, so that one can introduce what is needed at the appropriate time.
- This leads to "removing" control from the control problem.

Definition 1 Let M be a manifold.

(i) An ***affine subbundle*** of TM is a subset $\mathcal{A} \subset \mathsf{TM}$ with the property that for each $x \in \mathsf{M}$ there exists a neighbourhood $\mathcal{U}$ of x and vector fields $X_0, X_1, \dots, X_k$ defined on $\mathcal{U}$ for which

$$\mathcal{A}_x \triangleq \mathcal{A} \cap \mathsf{T}_x\mathsf{M} = \left\{ X_0(x) + \sum_{a=1}^{k} u^a X_a(x) \;\middle|\; u \in \mathbb{R}^k \right\}, \qquad x \in \mathcal{U}.$$

(ii) An ***affine system*** in an affine subbundle $\mathcal{A}$ is an assignment to each $x \in \mathsf{M}$ a subset $\mathscr{A}(x) \subset \mathcal{A}_x$.

(iii) A ***trajectory*** of an affine system $\mathscr{A}$ is a locally absolutely continuous $\gamma\colon I \to \mathsf{M}$ satisfying $\gamma'(t) \in \mathscr{A}(\gamma(t))$, a.e. $t \in I$. •

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$\mathcal{A}$-variations

- For simplicity, take $\mathscr{A}(x) = \mathcal{A}_x$ for each $x \in \mathsf{M}$.
- Given a reference trajectory γ_{ref}, what should define the linearisation?

Definition 2 Let $\gamma_{\mathrm{ref}} \colon I \to \mathsf{M}$ be a trajectory for an affine system $\mathcal{A}$. An ***$\mathcal{A}$-variation*** of γ_{ref} is a map $\sigma \colon I \times J \to \mathsf{M}$ with

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(i) some regularity properties,

(ii) for which $t \mapsto \sigma(t, s)$ is a trajectory of $\mathcal{A}$ for each $s \in J$, and for which

(iii) $\sigma(t, 0) = \gamma_{\mathrm{ref}}(t)$ for each $t \in I$. •

- For a variation σ define a vector field V_σ along γ_{ref} by

$$V_\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \sigma(s, t).$$

The geometry of $\mathcal{A}$-variations

- We wish to characterise variations geometrically. Suppose that γ_{ref} is an integral curve of some time-varying $\mathcal{A}$-valued vector field X_{ref} on M (as will be the case in practice).
- Given a vector field X on M, the **complete lift** of X, denoted by X^T, is the vector field on TM defined by $X^T(v_x) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} T_x\Phi^X_{0,s}(v_x)$.

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- Given $X \in \mathsf{T}_x\mathsf{M}$, the **vertical lift** of X through $v_x \in \mathsf{T}_x\mathsf{M}$, denoted $\mathrm{vlft}_{v_x}(X) \in \mathsf{T}_{v_x}\mathsf{TM}$, is the image of X under the natural isomorphism between $\mathsf{T}_x\mathsf{M}$ and $\mathsf{T}_{v_x}(\mathsf{T}_x\mathsf{M}) \subset \mathsf{T}_{v_x}\mathsf{TM}$.
- Define an affine subbundle on TM by

$$\mathcal{A}^T_{\mathsf{ref},v_x} = \{X^T_{\mathsf{ref}}(v_x) + \mathrm{vlft}_{v_x}(X) \mid X \in L(\mathcal{A})_x\},$$

where $L(\mathcal{A})$ is the linear part of $\mathcal{A}$.

Proposition 3 *For a reference trajectory γ_{ref} an integral curve of X_{ref}, and a vector field Υ along γ_{ref}, the following are equivalent:*

(i) *Υ is a trajectory for the affine system $\mathcal{A}^T_{\mathrm{ref}}$;*

(ii) *there exists an $\mathcal{A}$-variation σ of γ_{ref} for which $\Upsilon(t) = V_\sigma(t)$.*

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- *Punchline:* The linearisation of an affine system on M is a "linear" affine system on TM. (One can generally talk about linear systems defined on vector bundles.)
- This answers the question, "Where does the linearisation live?"
- It also makes not so obvious the answers to all the other questions we asked that follow this.

Controllability

- Define reachable sets: let $\mathsf{Traj}(\mathcal{A})$ denote the set of trajectories for $\mathcal{A}$ and let $\mathsf{Traj}(\mathcal{A}^T_{\mathsf{ref}})$ denote the set of trajectories of $\mathcal{A}^T_{\mathsf{ref}}$.
- Suppose that $\gamma_{\mathsf{ref}}(t_0) = x_0$.
- Then write

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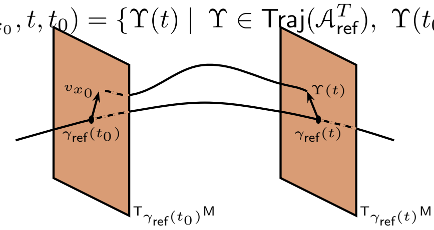
$$\mathcal{R}_{\mathcal{A}}(x_0, t, t_0) = \{\gamma(t) \mid \gamma \in \mathsf{Traj}(\mathcal{A}),\ \gamma(t_0) = x_0\}$$
$$\mathcal{R}_{\mathcal{A}^T_{\mathsf{ref}}}(v_{x_0}, t, t_0) = \{\Upsilon(t) \mid \Upsilon \in \mathsf{Traj}(\mathcal{A}^T_{\mathsf{ref}}),\ \Upsilon(t_0) = v_{x_0}\}.$$



Definition 4 $\mathcal{A}$ is

(i) ***controllable at*** $\mathbf{t_0}$ along γ_{ref} if $\gamma_{\mathrm{ref}}(t) \in \mathrm{int}(\mathcal{R}_{\mathcal{A}}(x_0, t, t_0))$ for each $t > t_0$, and is

(ii) ***linearly controllable at*** $\mathbf{t_0}$ along γ_{ref} if $\mathcal{R}_{\mathcal{A}^T_{\mathrm{ref}}}(0_{x_0}, t, t_0) = \mathsf{T}_{\gamma_{\mathrm{ref}}(t)}M$ for each $t > t_0$. •

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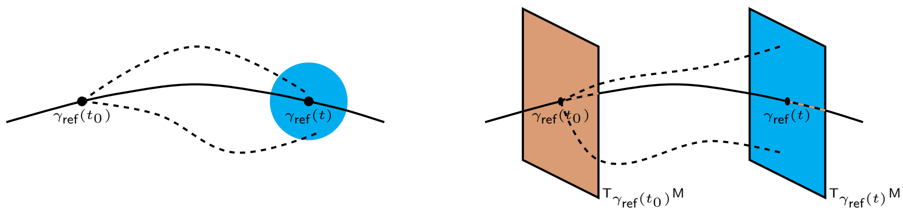

Controllable

Linearly controllable

Geometric characterisation of controllability

- Want the analogue of "smallest A-invariant subspace containing $\mathrm{image}(B)$."
- Define an operator $\mathscr{L}^{X_{\mathsf{ref}},\gamma_{\mathsf{ref}}}$ on the set of vector fields along γ_{ref} by

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$$\mathscr{L}^{X_{\mathsf{ref}},\gamma_{\mathsf{ref}}}(V_{\gamma_{\mathsf{ref}}})(t) = [X_{\mathsf{ref},t}, V](\gamma_{\mathsf{ref}}(t)), \text{ a.e. } t \in I,$$

where V is a vector field on M and $V_{\gamma_{\mathsf{ref}}}$ is the section of $\mathsf{T}M$ along γ_{ref} defined by $V_{\gamma_{\mathsf{ref}}}(t) = V(\gamma_{\mathsf{ref}}(t))$.

- Denote by $\langle \mathscr{L}^{X_{\mathsf{ref}},\gamma_{\mathsf{ref}}}, L(\mathcal{A})_{t_0}\rangle$ the smallest $\mathscr{L}^{X_{\mathsf{ref}},\gamma_{\mathsf{ref}}}$-invariant distribution along γ_{ref} that agrees with $L(\mathcal{A})$ at $\gamma_{\mathsf{ref}}(t_0)$.

Theorem 5 *Let $\gamma_{\mathrm{ref}}\colon I \to M$ be a differentiable reference trajectory that is an integral curve for X_{ref}. For $t_0 \in I$ and $t > t_0$, the following sets are equal:*

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(i) $\mathcal{R}_{\mathcal{A}^T_{\mathrm{ref}}}(0_{x_0}, t, t_0)$;

(ii) $\mathrm{span}_{\mathbb{R}}\Big(\bigcup_{\substack{\tau\in[t_0,t]\\ v_\tau\in L(\mathcal{A})_{\gamma_{\mathrm{ref}}(\tau)}}} \Phi^{X^T_{\mathrm{ref}}}_{\tau,t}(v_\tau)\Big)$;

(iii) $\langle \mathscr{L}^{X_{\mathrm{ref}},\gamma_{\mathrm{ref}}}, L(\mathcal{A})_{t_0}\rangle_{\gamma_{\mathrm{ref}}(t)}$.

Future work

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- Stability and stabilisation.
- Quadratic optimal control; what is the geometric analogue of the Riccati equation?
- Go back to the mechanical setup and understand the special geometry there.