Removing "control" from control theory

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Motivation

- Usually in control theory we study **control-affine systems** of the form

$$\dot{x} = f_0(x) + \sum_{a=1}^{m} u_a f_a(x),$$

with f_0 the **drift vector field** and $f_1, \dots, f_m$ the **control vector fields**.

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- The choice of f_0 and $f_1, \dots, f_m$ is much like a choice of coordinates for a manifold:
  1. in a given example, there is often a natural choice;
  2. to do general theoretical developments, it is advantageous to develop a theory that is coordinate-independent.
- A "coordinate-independent" theory for control-affine systems is not fully, or even really initially, developed. In this talk, we indicate the first steps concerning how to do this.

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- Note that the "change of coordinates" for a control-affine system looks like

$$f_0 \mapsto \tilde{f}_0 = f_0 + \lambda^a f_a,$$
$$(f_1, \dots, f_m) \mapsto (\tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}), \quad \tilde{f}_\alpha = \Lambda^a_\alpha f_a$$

- We also note that the essential geometric object here is the affine subbundle $\mathcal{A}$ defined by

$$\mathcal{A}_x = \Big\{ f_0(x) + \sum_{a=1}^m u_a f_a(x) \;\Big|\; u \in \mathbb{R}^m \Big\}.$$

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Affine subbundles and affine systems

- We let M be an analytic manifold. A subset $\mathcal{A}$ is an **affine subbundle** if, for each $x \in \mathsf{M}$, there exists a neighbourhood $\mathcal{N}$ of x and vector fields $X_0, X_1, \dots, X_k$ on $\mathcal{N}$ such that

$$\mathcal{A}_x = \mathcal{A} \cap \mathsf{T}_x\mathsf{M} = \Big\{ X_0(x) + \sum_{j=1}^k u_j X_j(x) \;\Big|\; u \in \mathbb{R}^k \Big\}.$$

- Denote by $L(\mathcal{A})$ the distribution defined by $L(\mathcal{A})_x$ being the linear part of the affine subspace $\mathcal{A}_x$.
- An **affine system** $\mathscr{A}$ in $\mathcal{A}$ assigns to each $x \in \mathsf{M}$ a subset $\mathscr{A}(x) \subset \mathcal{A}_x$ such that
  1. $\mathrm{aff}(\mathscr{A}(x)) = \mathcal{A}_x$ and
  2. some regularity conditions are satisfied on the manner in which the sets $\mathscr{A}(x)$ change as x changes (e.g., to ensure nice properties for the trajectories as defined below).

- Some control theoretic concepts:
  1. A **trajectory** for $\mathscr{A}$ is a locally absolutely continuous curve $\xi\colon I \to \mathsf{M}$ such that $\xi'(t) \in \mathscr{A}(\xi(t))$ for a.e. $t \in I$. Let $\mathsf{Traj}_T(\mathscr{A})$ denote the trajectories for $\mathscr{A}$ for which $I = [0, T]$.
  2. A **state feedback** for $\mathscr{A}$ is a vector field X such that $X(x) \in \mathscr{A}(x)$ for each $x \in \mathsf{M}$.

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- Note that "control" has disappeared, as promised in the title. After all, controls are nothing more than "coordinates" for a specific choice of generators for $\mathcal{A}$.
- **Big objective:** *Using only the geometry of $\mathcal{A}$*, do control theory: controllability, stabilisability and stabilisation, optimal control, etc.
- We shall think a little about controllability and stabilisability. How does one even *define* these concepts only in terms of $\mathcal{A}$?

Controllability and stabilisability definitions

- $\mathcal{R}_{\mathscr{A}}(x_0, T) = \{\xi(T) \mid \xi \in \mathsf{Traj}_T(\mathscr{A}),\ \xi(0) = x_0\}$ and $\mathcal{R}_{\mathscr{A}}(x_0, \leq T) = \cup_{t \in [0,T]} \mathcal{R}_{\mathscr{A}}(x_0, t)$.
- An affine system $\mathscr{A}$ in $\mathcal{A}$ is **proper** at x_0 if $0_{x_0} \in \operatorname{int}(\operatorname{conv}(\mathscr{A}(x_0)))$.

Definition 1 Let $\mathcal{A}$ be an affine subbundle on M and let $x_0 \in \mathsf{M}$.

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(i) An affine system $\mathscr{A}$ in $\mathcal{A}$ is ***small-time locally controllable*** (***STLC***) from x_0 if there exists $T > 0$ such that $x_0 \in \operatorname{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq t))$ for each $t \in]0, T]$.

(ii) $\mathcal{A}$ is ***properly small-time locally controllable*** (***PSTLC***) from x_0 if an affine system $\mathscr{A}$ in $\mathcal{A}$ is STLC from x_0 whenever $\mathscr{A}$ is proper at x_0.

(iii) $\mathcal{A}$ is ***small-time locally uncontrollable*** (***STLCUC***) from x_0 if an affine system $\mathscr{A}$ in $\mathcal{A}$ is not STLC from x_0 whenever $\mathscr{A}(x_0)$ is compact.

(iv) $\mathcal{A}$ is ***conditionally small-time locally controllable*** (***CSTLC***) from x_0 if it is not PSTLC from x_0, but there exists an affine system $\mathscr{A}$ in $\mathcal{A}$, with $\mathscr{A}(x_0)$ compact, that is STLC from x_0. •

- It is not obvious why these are the right definitions, so let us make some comments on them.
  1. First of all, the notions of PSTLC, STLCUC, and CSTLC are applied to affine subbundles (not affine systems) as desired.
  2. It seems reasonable (and is true) that if $0_{x_0} \notin \operatorname{conv}(\mathscr{A}(x_0))$, then $\mathscr{A}$ is not STLC. Therefore, the definition of PSTLC roughly says that, "If an affine system in $\mathcal{A}$ has a chance to be STLC (i.e., $\mathscr{A}$ is proper at x_0), then it is."
  3. The reason for compactness in the definitions for STLCUC and CSTLC is the following. If $\mathscr{A}(x_0) = \mathcal{A}_{x_0}$ then $\mathscr{A}$ is STLC from x_0 if the involutive closure of $L(\mathcal{A})$ has maximal rank at x_0. (In more common language, this means that the involutive closure of the control vector fields is full rank.) Typically systems achieving controllability in this way require "large" inputs to overcome the effects of drift. We do not wish to call such systems PSTLC, but they will fall into the CSTLC category.

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- For stabilisability, the smoothness of the state feedback is critical.

Definition 2 Let $r \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$, let $\mathcal{A}$ be an affine subbundle on M, and let $x_0 \in \mathsf{M}$.

(i) An affine system $\mathscr{A}$ in $\mathcal{A}$ is ***C^r-locally asymptotically stabilisable*** (**LAS^r**) to x_0 if there exists a neighbourhood $\mathcal{N}$ of x_0 and a C^r-state feedback X such that x_0 is an asymptotically stable equilibrium point for $X|\mathcal{N}$.

(ii) $\mathcal{A}$ is ***C^r-properly locally asymptotically stabilisable*** (**PLAS^r**) to x_0 if an affine system $\mathscr{A}$ in $\mathcal{A}$ is LAS^r whenever $\mathscr{A}$ is proper at x_0.

(iii) $\mathcal{A}$ is ***C^r-locally asymptotically unstabilisable*** (**LAUS^r**) to x_0 if an affine system $\mathscr{A}$ in $\mathcal{A}$ is not LAS^r to x_0 whenever $\mathscr{A}(x_0)$ is compact.

(iv) $\mathcal{A}$ is ***C^r-conditionally locally asymptotically stabilisable*** (**CLAS^r**) to x_0 if it is not PLAS^r to x_0, but there exists an affine system $\mathscr{A}$ in $\mathcal{A}$, with $\mathscr{A}(x_0)$ compact, that is LAS^r to x_0. •

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- These definitions can be, and probably should be, refined further by additionally considering "almost C^r-state feedback" (meaning that the feedback is of class C^r on a punctured neighbourhood of x_0).
- When talking about C^0-local asymptotic stabilisability, one must be careful about matters concerning uniqueness of solutions.
- The motivation for these definitions is much like that for the controllability definitions.
- Here's an example of a system that is CLAS$^\omega$: $\dot{x} = x + xu$; $x, u \in \mathbb{R}$.

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- The matter of stabilisability to x_0 is known to be closely tied to that of the ability to steer, in an open-loop manner, points in a neighbourhood of x_0 to x_0.[1,2] This is called "asymptotic controllability."
- A little more precisely, what can be shown is that if a system is asymptotically controllable to x_0, then it is locally asymptotically stabilisable, possibly using feedback that is discontinuous.
- This leads us to be careful about defining asymptotic controllability for affine subbundles. The definitions take a by now familiar form.

[1] Clarke, Ledyaev, Sontag, and Subotin, *IEEE Trans. Automat. Control*, **42**(10), 1394–1407, 1997

[2] Ancona and Bressan, *ESAIM Control Optim. Calc. Var.*, **4**, 445–471, 1999

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Definition 3 Let $\mathcal{A}$ be an affine subbundle on M and let $x_0 \in \mathsf{M}$.

(i) An affine system $\mathscr{A}$ in $\mathcal{A}$ is ***locally asymptotically controllable*** (***LAC***) to x_0 if there exists a neighbourhood $\mathcal{N}$ of x_0 such that, for each $x \in \mathcal{N}$, there exists a trajectory $\xi\colon [0, \infty[\to \mathsf{M}$ for $\mathscr{A}$ such that $\xi(0) = x$ and $\lim_{t\to\infty} \xi(t) = x_0$.

(ii) $\mathcal{A}$ is ***properly locally asymptotically controllable*** (***PLAC***) to x_0 if an affine system $\mathscr{A}$ in $\mathcal{A}$ is LAC whenever $\mathscr{A}$ is proper at x_0.

(iii) $\mathcal{A}$ is ***locally asymptotically uncontrollable*** (***LAUC***) to x_0 if an affine system $\mathscr{A}$ in $\mathcal{A}$ is not LAC whenever $\mathscr{A}(x_0)$ is compact.

(iv) $\mathcal{A}$ is ***conditionally locally asymptotically controllable*** (***CLAC***) to x_0 if it is not PLAC to x_0, but there exists an affine system $\mathscr{A}$, with $\mathscr{A}(x_0)$ compact, that is LAC to x_0. •

The big job ahead

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Characterise all these notions of controllability and stabilisability, and the relationships between them.

Almost nothing has been done in this area.

Some "simple" controllability theorems

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- The first result is a "zeroth-order" condition.[1]

Theorem 1 *Let $\mathcal{A}$ be an affine subbundle on M and let $x_0 \in \mathsf{M}$.*

(i) If $\mathcal{A}_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$, then $\mathcal{A}$ is PSTLC from x_0.

(ii) If $0_{x_0} \notin \mathcal{A}_{x_0}$, or equivalently if $L(\mathcal{A})_{x_0} \neq \mathcal{A}_{x_0}$, then $\mathcal{A}$ is STLUC from x_0.

- This result is "obvious," on a moment's thought.

[1] Sussmann, *SIAM J. Control Optim.*, **16**(5), 790–802, 1978

- Our next result needs some notation.
  1. $Z_{x_0}(\mathcal{A}) = \{X \in \Gamma(\mathcal{A}) \mid\ X(x_0) = 0_{x_0}\}$.
  2. For $\mathscr{L} \subset \mathsf{End}(\mathsf{T}_{x_0}\mathsf{M})$ and S_{x_0} a subspace of $\mathsf{T}_{x_0}\mathsf{M}$, $\langle \mathscr{L}, \mathsf{S}_{x_0}\rangle$ denotes the smallest subspace of $\mathsf{T}_{x_0}\mathsf{M}$ that (a) contains S_{x_0} and (b) is invariant under each element of $\mathscr{L}$.

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  3. For $k \in \mathbb{N}$, let $\mathsf{Lie}^{(k)}(\mathcal{A})$ be the distribution generated by $\mathcal{A}$-valued vector fields and their brackets of degree up to k. Thus

$$\mathsf{Lie}^{(1)}(\mathcal{A})_x = \operatorname{span}_{\mathbb{R}}(X(x) \mid\ X \in \Gamma(\mathcal{A})),$$
$$\mathsf{Lie}^{(2)}(\mathcal{A})_x = \mathsf{Lie}^{(1)}(\mathcal{A})_x + \operatorname{span}_{\mathbb{R}}([X_1, X_2](x) \mid\ X_1, X_2 \in \Gamma(\mathcal{A})).$$

- If $X \in Z_{x_0}(\mathcal{A})$, then define $L_X \in \mathsf{End}(\mathsf{T}_{x_0}\mathsf{M})$ by $L_X(v) = [X, V](x_0)$. This gives an inclusion of $Z_{x_0}(\mathcal{A})$ in $\mathsf{End}(\mathsf{T}_{x_0}\mathsf{M})$.

- The next condition is a "first-order" condition.[1]

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Theorem 2 *Let $\mathcal{A}$ be an affine subbundle on M and let $x_0 \in \mathsf{M}$. If*

$$\left\langle Z_{x_0}(\mathcal{A}), \mathrm{Lie}^{(2)}(\mathcal{A})_{x_0}\right\rangle = \mathsf{T}_{x_0}\mathsf{M},$$

then $\mathcal{A}$ is PSTLC from x_0.

[1] Bianchini and Stefani, *Internat. J. Control*, **39**(4), 701–714, 1984

Two second-order results

- As motivation, consider the following system:

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$$\dot{x}_1 = u_1,$$
$$\dot{x}_2 = u_2,$$
$$\dot{x}_3 = x_1^2 + \alpha x_2^2.$$

It is intuitively clear that this system is STLC from $(0,0,0)$ if and only if $\alpha < 0$.

- We wish to generalise this example.

Vector-valued quadratic forms

- $\mathsf{TS}^2(\mathsf{V};\mathsf{W})$: set of symmetric bilinear maps from vector space V to vector space W.
- For $\lambda \in \mathsf{W}^*$, define $\lambda B \in \mathsf{TS}^2(\mathsf{V};\mathbb{R})$ by $\lambda B(v_1, v_2) = \langle \lambda; B(v_1, v_2)\rangle$.

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Definition 4 $B \in \mathrm{TS}^2(\mathsf{V};\mathsf{W})$ is

(i) ***definite*** if there exists $\lambda \in \mathsf{W}^*$ such that λB is positive-definite, and is

(ii) ***essentially indefinite*** if, for each $\lambda \in W^*$, λB is either zero or neither positive nor negative semidefinite. •

Vector-valued quadratic forms for affine subbundles

- *Assume that x_0 is a regular point for $L(\mathcal{A})$.* It is possible, although complicated, to proceed without this assumption.
- Given an affine subbundle $\mathcal{A}$ on M, a point $x_0 \in \mathsf{M}$ for which $0_{x_0} \in \mathcal{A}_{x_0}$, and a subspace $\mathsf{S}_{x_0} \subset \mathsf{T}_{x_0}\mathsf{M}$, define $B_{\mathcal{A}}(\mathsf{S}_{x_0}) \in \mathsf{TS}^2(L(\mathcal{A})_{x_0}; \mathsf{T}_{x_0}\mathsf{M}/\mathsf{S}_{x_0})$ by

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$$B_{\mathcal{A}}(\mathsf{S}_{x_0})(v_1, v_2) = \pi_{\mathsf{S}_{x_0}}([V_1, [X, V_2]](x_0)),$$

where

1. $X \in Z_{x_0}(\mathcal{A})$,
2. $\pi_{\mathsf{S}_{x_0}} : \mathsf{T}_{x_0}\mathsf{M} \to \mathsf{T}_{x_0}\mathsf{M}/\mathsf{S}_{x_0}$ is the canonical projection, and
3. V_1 and V_2 are vector fields extending $v_1, v_2 \in L(\mathcal{A})_{x_0}$.

- It is not at all clear that $B_{\mathcal{A}}(\mathsf{S}_{x_0})$ is well-defined, and indeed it is only defined if S_{x_0} has certain properties (that will hold in the theorem statements below) and if our assumption on the regularity of $L(\mathcal{A})$ at x_0 holds.

Controllability theorems

- Our first theorem gives conditions for $\mathcal{A}$ to be PSTLC.

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Theorem 3 *Let $\mathcal{A}$ be an affine subbundle on M, let $x_0 \in \mathsf{M}$, and let $\mathsf{S}_{x_0} = \langle Z_{x_0}(\mathcal{A}), \mathrm{Lie}^{(2)}(\mathcal{A})_{x_0}\rangle$. If*

(i) $0_{x_0} \in \mathcal{A}_{x_0}$,

(ii) $\langle Z_{x_0}(\mathcal{A}), \mathrm{Lie}^{(3)}(\mathcal{A})_{x_0}\rangle$, *and*

(iii) $B_{\mathcal{A}}(\mathsf{S}_{x_0})$ *is essentially indefinite,*

then $\mathcal{A}$ is PSTLC.

- The next result gives conditions for $\mathcal{A}$ to be STLCUC.

Theorem 4 *Let $\mathcal{A}$ be an affine subbundle on* M*, let* $x_0 \in \mathsf{M}$*, and let* $\mathsf{S}_{x_0} = \langle Z_{x_0}(\mathcal{A}), L(\mathcal{A})_{x_0}\rangle + \mathrm{Lie}^{(\infty)}(L(\mathcal{A}))_{x_0}$*. If*

(i) $0_{x_0} \in \mathcal{A}_{x_0}$*,*

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(ii) x_0 *is a regular point for* $\mathrm{Lie}^{(\infty)}(L(\mathcal{A}))_{x_0}$*, and*

(iii) $B_{\mathcal{A}}(\mathsf{S}_{x_0})$ *is definite,*

then $\mathcal{A}$ is STLCUC.

- Not only are these theorems difficult to prove, they are not obvious. For example, the matter of the choices of S_{x_0} in each case is crucial.

Open problems

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- There are many, and at this point they are very general.
  1. Higher-order conditions for controllability.
  2. Relationship between controllability *from* a point with stabilisability *to* a point.
  3. Other parts of control theory:
     (a) linearisation;
     (b) optimal control;
     (c) design of feedback control laws;
     (d) etc.