Problems and Challenges in Control Theory (for Mechanical Systems)

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What is control theory?

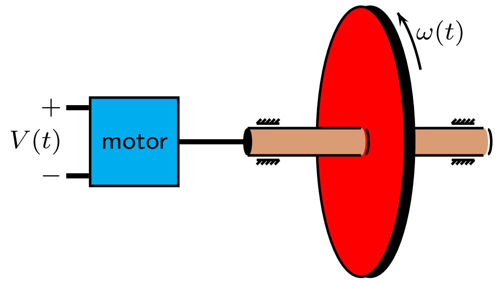
- Control theory is the study of the manipulation of dynamical processes to achieve desired objectives.
- There are many sorts of models considered in control theory. Let us fix one:

$$\dot{x}(t) = f(x(t), u(t)),$$
$$y(t) = g(x(t), u(t)).$$

1. The variable $x \in \mathbb{R}^n$ is the ***state*** whose behaviour is being controlled.
2. The variable $u \in \mathbb{R}^m$ is the ***control*** or ***input*** which we can specify as we like.
3. The variable $y \in \mathbb{R}^p$ is the ***output***, and might represent what can be measured.

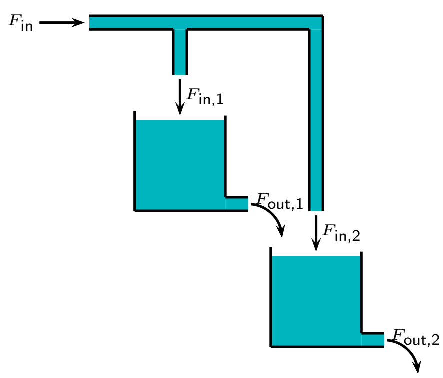
Simple examples

- Servo-motor:



1. State: Angular position and velocity of flywheel (and possibly armature current in motor, for accurate model).
2. Input: Voltage $V(t)$.
3. Output: Angular velocity of flywheel.
4. Typical control objective: Design $V(t)$ so that a desired flywheel angular velocity $\omega(t)$ is achieved and maintained.

- Coupled tanks:



1. States: Heights $h_1(t)$ and $h_2(t)$ of water in tanks.
2. Input: Input flow $F_{\text{in}}(t)$.
3. Output: One or both tank heights.
4. Typical control objective: Design $F_{\text{in}}(t)$ so that desired heights $h_1(t)$ and $h_2(t)$ are achieved and maintained.

Typical control problems

1. ***Stabilisation:*** Design the control u_{des} (as a function of state, or time, or possibly both) so that a desired state x_0 is a stable equilibrium for the differential equation

$$\dot{x}(t) = f(x(t), \underbrace{u_{\mathsf{des}}(x,t)}_{\substack{\text{or } u_{\mathsf{des}}(t) \\ \text{or } u_{\mathsf{des}}(x)}}).$$

2. ***Output tracking:*** Obtain u_{des} so that the output follows a specified trajectory $y_{\mathsf{des}}(t)$.

3. ***Motion planning:*** Steer the state from x_1 to x_2.

4. ***Optimal control:*** Do any of the above while minimising some cost function, e.g., time, or "control energy."

Typical approaches

- Linear systems:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

  1. Much—a huge amount—has been done using tools from linear algebra and functional analysis.
  2. This is what is typically used in practice.

- Nonlinear systems:
  1. Linearise—this works sometimes.
  2. In general, nonlinear control theory is very difficult.
  3. Need to focus on structure to gain understanding.

Geometric control theory

- Let us specialise a little to systems that are ***control-affine***:

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u_a(t) f_a(x(t)), \qquad (1)$$

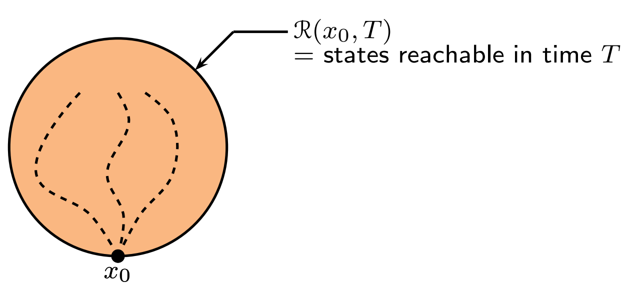
  where f_0 is the ***drift vector field*** and $f_1, \dots, f_m$ are the ***control vector fields***.

- Forget about outputs to keep things simple.

- In ***geometric control theory*** we are interested in understanding the properties of system (1) from the point of view of differential geometry.

- I will not assume you know any differential geometry...

Reachable sets



$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u_a(t) f_a(x(t))$$

- Also define $\mathcal{R}(x_0, \leq T) = \cup_{t \in [0,T]} \mathcal{R}(x_0, t)$ = states reachable in time at most T.

- *Question:* Why is the reachable set important?
- *Answer:*
  1. It gives some idea of what is possible as far as control objectives. For example, maybe the reachable set can tell you that it is not possible to steer between states x_1 and x_2.
  2. Properties of the reachable set appear (although often are hidden) as hypotheses in many design procedures.
  3. There are non-obvious and important connections between the reachable set and optimal control theory.
  4. There should be (as yet unexplored) relationships between the reachable set and the stabilisation problem.

Exploring the reachable set

- The definition of the reachable set is not useful, because to define it requires computing solutions to differential equations—this is impossible in general.
- *Question:* Are there *computable* ways of characterising the reachable set?
- Consider the following simple control system:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x).$$

- Apply the control

$$u(t) = \begin{cases} (1,0), & 0 \le t < T, \\ (0,1), & T \le t < 2T, \\ (-1,0), & 2T \le t < 3T, \\ (0,-1), & 3T \le t \le 4T. \end{cases}$$

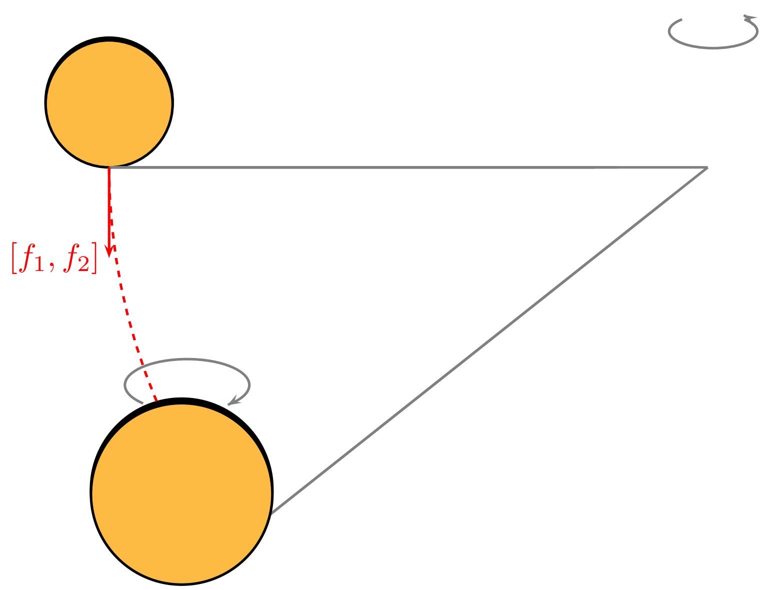
- Where does $x(4T)$ end up? (Note that $x(4T)$ is clearly in the reachable set.)
- We determine (e.g., by Taylor expansion) that

$$x(4T) = x(0) + T^2\,[f_1, f_2](x(0)) + \dots, \qquad [f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2.$$

- $[f_1, f_2]$ is the ***Lie bracket*** of f_1 and f_2.
- By applying suitable controls to our general system, one may move in the directions

$$\begin{aligned}
&f_0,\ f_1, \dots, f_m,\\
&[f_a, f_b], \qquad a, b = 0, \dots, m,\\
&[f_a, [f_b, f_c]], \qquad a, b, c = 0, \dots, m,\\
&\text{etc.}
\end{aligned}$$

A mechanical exhibition of the Lie bracket



A theorem on the nature of the reachable set

Theorem 1

Procedure: *Compute Lie brackets*

$$[f_{a_1}, [f_{a_2}, [\dots, [f_{a_{k-1}}, f_{a_k}]]]], \ k \in \mathbb{Z}_+, \ a_1, \dots, a_k \in \{0, 1, \dots, m\}.$$

Check: *At some point in the computation, do some collection of these Lie brackets evaluated at x_0 form a basis?*
Conclusion: *If so,* $\mathrm{int}(\mathcal{R}(x_0, \leq T) \neq \varnothing$.

Is the Lie bracket important?

- Our previous analysis and theorem suggest that the Lie bracket is interesting in geometric control theory.
- It is also important in differential geometry, physics, some areas of partial differential equations.
- It is also an example of a general structure in algebra known as a ***Lie algebra***. Associated with these are ***Lie groups***.
- Thus Lie brackets appear in various contexts in mathematics, as well as being essential in geometric control theory.

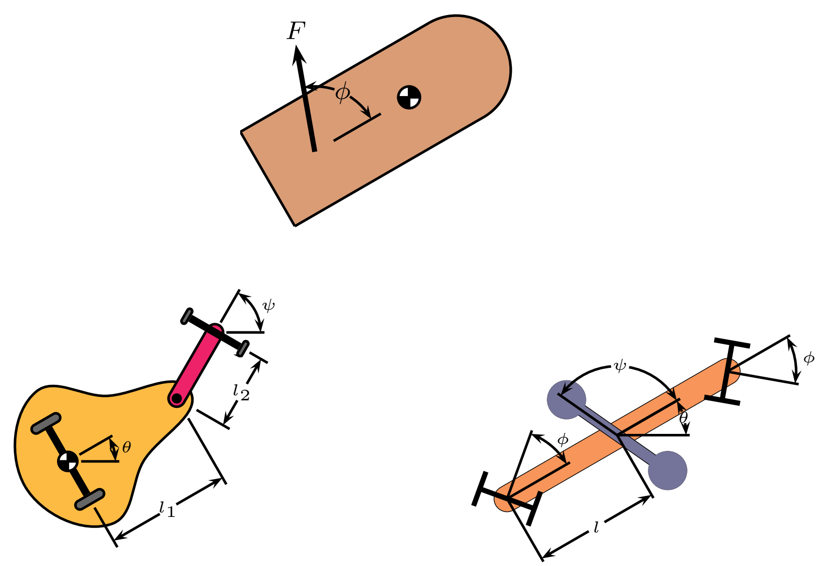
Refining the reachable set

Accessibility

Controllability

- Accessibility is essentially exactly characterised by the previous theorem.
- Controllability is "impossible" (precisely, it is NP-hard, in the language of computational complexity).

Mechanical systems

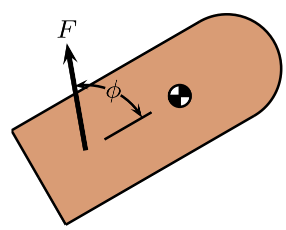


Geometric mechanics

- Apart from control theory, mechanics has its own very interesting mathematical structure.
- The most important part of the physical model is the kinetic energy
  ⟶ leads to an interesting geometric structure called a ***Riemannian metric***.
- Also interesting are ***nonholonomic constraints*** (as in last two of the examples).
- The structure associated with a nonholonomic constraint is a ***distribution***
  ⟶ relationship between Riemannian metric and constraint distribution gives lots of interesting problems.

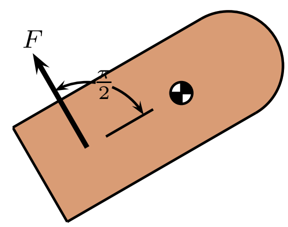
A simple example in detail

- Hovercraft system:



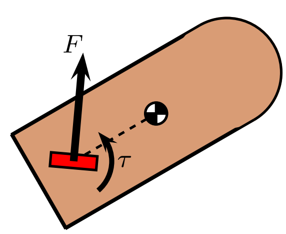
1. *Question:* Is the system accessible?
2. *Answer:* Yes (easy).
3. *Question:* Is the system controllable?
4. *Answer:* Yes (a little harder).
5. *Question:* Can we design an algorithm to steer from state to state?
6. *Answer:* Yes, if we are quite clever.
7. *Question:* Can we design an algorithm to stabilise a desired state?
8. *Answer:* Yes, but we do not understand this very well.

Make the example harder



1. *Question:* Is the system accessible?
2. *Answer:* Yes (easy).
3. *Question:* Is the system controllable?
4. *Answer:* No, at least not locally (nontrivial).
5. *Question:* Can we design an algorithm to steer from state to state?
6. *Answer:* Unknown.
7. *Question:* Can we design an algorithm to stabilise a desired state?
8. *Answer:* Unknown

Make the example different



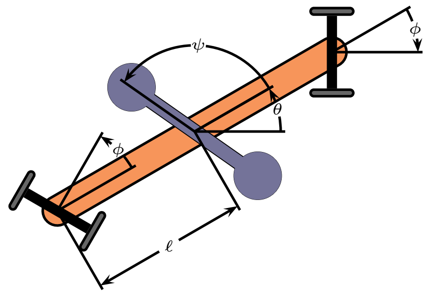
1. *Question:* Is the system accessible?
2. *Answer:* Yes (easy).
3. *Question:* Is the system controllable?
4. *Answer:* No, at least not locally (getting really difficult now).
5. *Question:* Can we design an algorithm to steer from state to state?
6. *Answer:* Unknown.
7. *Question:* Can we design an algorithm to stabilise a desired state?
8. *Answer:* Unknown

Punchline Even easy problems can be very difficult.

Motion planning for the easy planar body

- Movies for the planar body.

Nonholonomic mechanics: snakeboard example



- Snakeboard corporate movies.
- Snakeboard gaits.

Snakeboard motion planning

- The movies suggest that the snakeboard is controllable. It is.
- *Problem:* Can one design an algorithm to steer the snakeboard from a desired initial position to a desired final position?
- *Answer:* Yes!

But where's the mathematics?

- Control theory is widely practised as an engineering discipline.
- But it is also a mathematical subject in its own right.
- It has many branches.
- Linear control theory:
  1. linear differential equations;
  2. linear algebra;
  3. complex function theory;
  4. measure theory;
  5. functional analysis;
  6. operator theory;
  7. convex analysis.

- Nonlinear control theory:
  1. linear algebra;
  2. advanced differential equations;
  3. measure theory;
  4. "simple" differential geometry.
- Geometric control theory:
  1. linear algebra;
  2. advanced differential equations;
  3. measure theory;
  4. differential geometry;
  5. theory of distributions;
  6. analytic differential geometry (e.g., no partitions of unity);

- Control theory for mechanical systems:
  1. all the stuff from geometric control theory plus
  2. Riemannian geometry;
  3. affine differential geometry;
  4. Lie groups;
  5. some physics, if it interests you.
- *Important fact:* Each branch of control theory also has its own unique mathematical problems. That is, control theory *is* a subject in mathematics.

Summary

- Control theory is a broad subject, which uses a huge variety of mathematics, and possesses its own intricate mathematical problems.
- Mechanical control systems provide a class of systems with rich geometric structure.
- There is much work to be done here.

Acknowledgements for movies

1. The Snakeboard manufactured by Snakeboard USA, `http://www.snakeboard.com/`.
2. The Roller Racer is manufactured by Mason Corporation, `http://www.masoncorporation.com/`.
3. The Plasma Car is made by PlaSmart Corporation, `http://www.plasmacar.com/`.