# Problems and partial results in energy shaping

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## System definitions

- We will consider various flavours of systems, depending on whether they represent open-loop or closed-loop, or linear or nonlinear.

## Nonlinear system definitions

**Definition 1** A ***simple mechanical control system*** is a quadruple $(\mathsf{Q}, \mathbb{G}, V, \mathscr{F} = \{F^1, \dots, F^m\})$ where

(i) $\mathsf{Q}$ is an $n$-dimensional manifold,

(ii) $\mathbb{G}$ is a Riemannian metric on $\mathsf{Q}$,

(iii) $V$ is a function on $\mathsf{Q}$, and

(iv) $F^1, \dots, F^m$ are one-forms on $\mathsf{Q}$, generating a subbundle of $\mathsf{T}^*\mathsf{Q}$ which we denote by $\mathcal{F}$.

We assume all data to be at least of class $C^\infty$. •

- This will typically be the open-loop control system.

- The closed-loop system will be the following.

**Definition 2** A ***forced simple mechanical system*** is a quadruple $(\mathsf{Q}, \mathbb{G}, V, F)$ where

(i) $\mathsf{Q}$ is an $n$-dimensional manifold,

(ii) $\mathbb{G}$ is a Riemannian metric on $\mathsf{Q}$,

(iii) $V$ is a function on $\mathsf{Q}$, and

(iv) $F\colon \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q}$ is a bundle map over $\mathrm{id}_\mathsf{Q}$ called the ***external force***,

where we assume that all data is at least class $C^\infty$.

An external force $F$ is

(v) ***dissipative*** if $F(v_q) = -R^\flat(v_q)$, where $R$ is a symmetric positive-semidefinite $(0,2)$-tensor field called a ***Rayleigh dissipation tensor***, is

(vi) ***linearly gyroscopic*** if $F(v_q) = -C^\flat(v_q)$, where $C$ is a skew-symmetric $(0,2)$-tensor field called the ***linear gyroscopic tensor***, and is

(vii) ***quadratically gyroscopic*** if $F(v_q) = -B^\flat(v_q)$, where $B$ is a $(0,3)$-tensor field, called the ***quadratic gyroscopic tensor***, satisfying $B(u_q, v_q, w_q) = -B(v_q, u_q, w_q)$, for all $u_q, v_q, w_q \in \mathsf{TQ}$, and where

$$\langle B^\flat(v_q); u_q\rangle = B(u_q, v_q, v_q).$$ •

## Equations of motion for nonlinear systems

- For a simple mechanical control system $(\mathsf{Q}, \mathbb{G}, V, \mathscr{F} = \{F^1, \dots, F^m\})$, the governing equations are

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}^\sharp \circ \boldsymbol{d}V(\gamma(t)) + \sum_{a=1}^m u^a(t)\mathbb{G}^\sharp \circ F^a(\gamma(t))$$

$$\big(\ddot{q} + M^{-1}(q)C(q,\dot{q})\dot{q} = -M^{-1}(q)\boldsymbol{d}V(q) + M^{-1}(q)G(q)u\big),$$

where $\overset{\mathbb{G}}{\nabla}$ is the Levi-Civita connection associated with $\mathbb{G}$.

- For a forced simple mechanical system $(\mathsf{Q}, \mathbb{G}, V, F)$, the governing equations are

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}^\sharp \circ \boldsymbol{d}V(\gamma(t)) + \mathbb{G}^\sharp \circ F(\gamma'(t))$$

$$\big(\ddot{q} + M^{-1}(q)C(q,\dot{q})\dot{q} = -M^{-1}(q)\boldsymbol{d}V(q) + M^{-1}(q)F(q,\dot{q})\big).$$

## Linear system definitions

- The linear open-loop control systems will have the following form.

**Definition 3** A ***linear mechanical control system*** is a quadruple $(\mathsf{V}, M, K, F)$ where

(i) $\mathsf{V}$ is a finite-dimensional $\mathbb{R}$-vector space,

(ii) $M$ is an inner product on $\mathsf{V}$,

(iii) $K$ is a symmetric $(0, 2)$-tensor on $\mathsf{V}$, and

(iv) $F \in \mathrm{L}(\mathbb{R}^m; \mathsf{V}^*)$. •

- Linear closed-loop systems have the following form.

**Definition 4** A ***forced linear mechanical system*** is a quadruple $(\mathsf{V}, M, K, (F_1, F_2))$ where

(i) $\mathsf{V}$ is a finite-dimensional $\mathbb{R}$-vector space,

(ii) $M$ is an inner product on $\mathsf{V}$,

(iii) $K$ is a symmetric $(0, 2)$-tensor on $\mathsf{V}$, and

(iv) $F_1$ and $F_2$ are linear maps from $\mathsf{V}$ to $\mathsf{V}^*$ defining the ***external force***.

If $F_1 = 0$ and $F_2$ is symmetric and negative semidefinite, then the external force is ***dissipative***, and if $F_1 = 0$ and $F_2$ is skew-symmetric, then the external force is ***gyroscopic***. •

### Equations of motion for linear systems

- For a linear mechanical control system $(\mathsf{V}, M, K, F)$, the governing equations are

$$\ddot{x}(t) + M^{\sharp} \circ K^{\flat}(x(t)) = M^{\sharp} \circ F(u(t)).$$

- For a forced linear mechanical system $(\mathsf{V}, M, K, (F_1, F_2))$, the governing equations are

$$\ddot{x}(t) + M^{\sharp} \circ K^{\flat}(x(t)) = M^{\sharp} \circ F_1(x(t)) + M^{\sharp} \circ F_2(\dot{x}(t)).$$

## Problem formulation

### Energy shaping feedback for nonlinear systems

- Define $\Lambda_{\mathsf{cl}} = \mathbb{G}_{\mathsf{ol}}^{\flat} \circ \mathbb{G}_{\mathsf{cl}}^{\sharp}$.

**Definition 5** An ***energy shaping feedback*** for a simple mechanical $\Sigma_{\mathsf{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \mathscr{F})$ with closed-loop system $\Sigma_{\mathsf{cl}} = (\mathsf{Q}, \mathbb{G}_{\mathsf{cl}}, V_{\mathsf{cl}}, -R_{\mathsf{cl}}^{\flat} - C_{\mathsf{cl}}^{\flat} - B_{\mathsf{cl}}^{\flat})$ is given by $F\colon \mathsf{TQ} \to \mathcal{F}$ with $F = -F_{\mathsf{kin}} - F_{\mathsf{pot}} - F_{\mathsf{diss}} - F_{\mathsf{gyr}}$, where

(i) $F_{\mathsf{kin}}\colon \mathsf{TQ} \to \mathcal{F}$ has the property that

$$\mathbb{G}_{\mathsf{ol}}^{\sharp} \circ F_{\mathsf{kin}}(\gamma'(t)) = \overset{\mathbb{G}_{\mathsf{cl}}}{\nabla}_{\gamma'(t)}\gamma'(t) + \mathbb{G}_{\mathsf{cl}}^{\sharp} \circ B_{\mathsf{cl}}^{\flat}(\gamma'(t)) - \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t),$$

(ii) $F_{\mathsf{pot}}\colon \mathsf{Q} \to \mathcal{F}$ has the property that

$$F_{\mathsf{pot}}(\gamma(t)) = \Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V_{\mathsf{cl}}(\gamma(t)) - \boldsymbol{d}V_{\mathsf{ol}}(\gamma(t)),$$

(iii) $F_{\mathsf{diss}} \colon \mathsf{TQ} \to \mathcal{F}$ has the property that

$$F_{\mathsf{diss}}(\gamma'(t)) = \Lambda_{\mathsf{cl}} \circ R^\flat_{\mathsf{cl}}(\gamma'(t)),$$

(iv) $F_{\mathsf{gyr}} \colon \mathsf{TQ} \to \mathcal{F}$ has the property that

$$F_{\mathsf{gyr}}(\gamma'(t)) = \Lambda_{\mathsf{cl}} \circ C^\flat_{\mathsf{cl}}(\gamma'(t)).$$

•

## Energy shaping feedback for linear systems

**Definition 6** Let $\Sigma_{\mathsf{ol}} = (\mathsf{V}, M_{\mathsf{ol}}, K_{\mathsf{ol}}, F)$ be a linear mechanical control system. A ***linear energy shaping feedback*** for $\Sigma_{\mathsf{ol}}$ is a linear map $u \colon \mathsf{V} \oplus \mathsf{V} \to \mathbb{R}^m$ with the property that there exists $M_{\mathsf{cl}}, K_{\mathsf{shp}}, R_{\mathsf{shp}} \in \mathrm{TS}^2(\mathsf{V})$ and $C_{\mathsf{shp}} \in \mathrm{T}\bigwedge^2(\mathsf{V})$ such that $M_{\mathsf{cl}}$ is an inner product and such that

$$F \circ u(x, v) = -\Lambda_{\mathsf{cl}} \circ K^\flat_{\mathsf{shp}}(x) - \Lambda_{\mathsf{cl}} \circ R^\flat_{\mathsf{shp}}(v) - \Lambda_{\mathsf{cl}} \circ C^\flat_{\mathsf{shp}}(v),$$

where $\Lambda_{\mathsf{cl}} = M^\flat_{\mathsf{ol}} \circ M^\sharp_{\mathsf{cl}}$. •

## The central problem of energy shaping

### Problems 1

1. For a given open-loop system, determine the set of closed-loop systems.
2. Given a certain property for the closed-loop system, does there exist an energy shaping feedback for which the closed-loop system has this property (e.g., stability)? •

## Linear energy shaping[1]

- Given: The open-loop linear mechanical control system $\Sigma_{\mathrm{ol}} = (\mathsf{V}, M_{\mathrm{ol}}, K_{\mathrm{ol}}, F)$.
- Assume that the pair $(\underbrace{M_{\mathrm{ol}}^{\sharp} \circ K_{\mathrm{ol}}^{\flat}}_{A}, \underbrace{M_{\mathrm{ol}}^{\sharp} \circ F}_{B})$ is controllable. (If not, then restrict to the controllable subspace.)
- Define $E_{\Sigma_{\mathrm{ol}}}$ to be the collection of $A \in \mathrm{L}(\mathsf{V};\mathsf{V})$ satisfying
  1. $A = \underbrace{M_{\mathrm{ol}}^{\sharp} \circ K_{\mathrm{ol}}^{\flat} + M_{\mathrm{ol}}^{\sharp} \circ F \circ L}_{A+B\circ K}$ for some $L \in \mathrm{L}(\mathsf{V};\mathbb{R}^m)$, and
  2. $A$ is diagonalisable over $\mathbb{R}$.

[1] Zenkov, MTNS'02

**Proposition 1** *Let* $\Sigma = (\mathsf{V}, M_{\mathsf{ol}}, K_{\mathsf{ol}}, F)$ *be a linear mechanical control system. Then, for* $M_{\mathsf{cl}}, K_{\mathsf{cl}} \in \mathrm{TS}^2(\mathsf{V})$ *with* $M_{\mathsf{cl}}$ *an inner product, the following are equivalent:*

*(i)* *there exists a linear feedback* $u\colon \mathsf{V} \oplus \mathsf{V} \to \mathbb{R}^m$ *of the form* $x \oplus v \mapsto -L(x)$ *for which the dynamics of the closed-loop system are those of the forced linear mechanical system* $(\mathsf{V}, M_{\mathsf{cl}}, K_{\mathsf{cl}}, (0, 0))$*;*

*(ii)* $M_{\mathsf{cl}}^{\sharp} \circ K_{\mathsf{cl}}^{\flat} \in E_{\Sigma_{\mathsf{ol}}}$*.*

**Corollary 1** *A controllable linear mechanical control system can be stabilised by linear energy shaping feedback.*

- One can construct a multitude of explicit ways to "pull apart" $M_{\mathsf{cl}}^{\sharp} \circ K_{\mathsf{cl}}^{\flat}$ to yield $M_{\mathsf{cl}}$ and $K_{\mathsf{cl}}$.

## Some results on potential shaping

### The classical result[1]

- Given: The open-loop simple mechanical control system $\Sigma_{\mathsf{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \mathscr{F})$.
- Let $\mathcal{F}^{(\infty)}$ be the largest integrable codistribution contained in $\mathcal{F}$.
- Assume that $\mathcal{F}$ and $\mathcal{F}^{(\infty)}$ are regular.
- Let $C^\infty(\mathsf{Q})_{\mathcal{F}}$ denote the set of functions $f$ for which $\boldsymbol{d}f \in \Gamma^\infty(\mathcal{F}^{(\infty)})$.

**Proposition 2 (van der Schaft 1986)** *The difference between the closed- and open-loop potentials lies in* $C^\infty(\mathsf{Q})_{\mathcal{F}}$*.*

[1] van der Schaft, *Nonlinear Anal. TMA*, **10**(10), 1021–1035, 1986

## Potential shaping after kinetic shaping

- Now suppose that we have done some kinetic shaping to arrive at a closed-loop kinetic energy metric $\mathbb{G}_{\mathsf{cl}}$. Recall that $\Lambda_{\mathsf{cl}} = \mathbb{G}_{\mathsf{ol}}^{\flat} \circ \mathbb{G}_{\mathsf{cl}}^{\sharp}$.
- Let $\mathcal{F}_{\mathsf{cl}} = \Lambda_{\mathsf{cl}}(\mathcal{F})$ and let $\mathcal{F}_{\mathsf{cl}}^{(\infty)}$ be the largest integrable codistribution contained in $\mathcal{F}_{\mathsf{cl}}$.
- Assume that $\mathcal{F}_{\mathsf{cl}}$ and $\mathcal{F}_{\mathsf{cl}}^{(\infty)}$ are regular.
- Define

$$\mathrm{PS}(\mathsf{Q}) = \{V_{\mathsf{cl}} \in C^{\infty}(\mathsf{Q}) \mid \boldsymbol{d}V_{\mathsf{cl}} - \Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V_{\mathsf{ol}} \in \Gamma^{\infty}(\mathcal{F}_{\mathsf{cl}})\}$$

  and $L(\mathrm{PS}(\mathsf{Q})) = C^{\infty}(\mathsf{Q})_{\mathcal{F}_{\mathsf{cl}}}$.

**Proposition 3** $\mathrm{PS}(\mathsf{Q})$ *is an affine subspace (possibly empty) of* $C^{\infty}(\mathsf{Q})$ *modelled on the subspace* $L(\mathrm{PS}(\mathsf{Q}))$.

## Interpretation

- Recall the situation with the linear equation $Ax = b$:
  1. $b \notin \mathrm{image}(A)$: No solutions.
  2. $b \in \mathrm{image}(A)$: Set of solutions is an affine subspace modelled on $\ker(A)$.
- In the classical potential shaping case, $b = 0$ in the analogue.
- We do not yet understand conditions for the analogue of $b \notin \mathrm{image}(A)$ or $b \in \mathrm{image}(A)$, i.e., we do not understand the integrability of the potential shaping p.d.e.
- Note that the affine subspace is modelled on $C^{\infty}(\mathsf{Q})_{\mathcal{F}_{\mathsf{cl}}}$.
  1. This subspace might be trivial, even when the classical energy shaping subspace, $C^{\infty}(\mathsf{Q})_{\mathcal{F}}$, is not.
  2. If $\mathrm{codim}(\mathcal{F}) = 1$, then $\mathrm{codim}(\mathcal{F}_{\mathsf{cl}}) = 1$, and so Frobenius's Theorem guarantees that $C^{\infty}(\mathsf{Q})_{\mathcal{F}_{\mathsf{cl}}}$ is not trivial.

## Setting up the potential shaping problem for integrability tests[1]

- Think of the exterior derivative as a map, denoted by $\boldsymbol{d}_1$, from $\mathsf{J}^1(\mathcal{F}_{\mathsf{cl}})$ to $\mathrm{T}\bigwedge^2(\mathsf{TQ})$.
- Define

$$P_{\mathsf{PS}}(V_{\mathsf{ol}})_q = \{j^1 F_{\mathsf{cl}}(q) \mid \boldsymbol{d}_1(j^1 F_{\mathsf{cl}}(q)) = -\boldsymbol{d}(\Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V_{\mathsf{ol}})(q)\}.$$

**Proposition 4** *Suppose that the first cohomology group of* Q *is zero. Then a function* $V_{\mathsf{cl}}$ *is a possible closed-loop potential function if and only if* $\boldsymbol{d}V_{\mathsf{cl}} = F_{\mathsf{cl}} + \Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V_{\mathsf{ol}}$ *where* $F_{\mathsf{cl}}$ *is a section of* $\mathcal{F}_{\mathsf{cl}}$ *having the property that* $j^1 F_{\mathsf{cl}}$ *takes values in* $P_{\mathsf{PS}}(V_{\mathsf{ol}})$.

- This result puts the kinetic shaping problem in a form where the techniques of Spencer, Serre, Quillen, Goldschmidt, etc. are applicable.

[1] Spencer, *Ann. Math.*, **76**(3), 306–398 and 399–445, 1962,
Quillen, PhD thesis, Harvard University, 1964,
Serre, Appendix to Guillemin/Sternberg, *Bull. Amer. Math. Soc. (N.S.)*, **70**, 16–47, 1964
Goldschmidt, *J. Differential Geom.*, **1**, 269–307, 1967

# Formulations of the kinetic shaping problem

## An affine connection formulation

- For a general affine connection $\nabla$ and Riemannian metric $\mathbb{G}$ with its Levi-Civita connection $\overset{\mathbb{G}}{\nabla}$, define a $(0,3)$-tensor field $D_{\nabla,\mathbb{G}}$ by

$$\mathbb{G}(\nabla_X Y, Z) = \mathbb{G}(\overset{\mathbb{G}}{\nabla}_X Y, Z) + D_{\nabla,\mathbb{G}}(Z, X, Y).$$

- For a $(0,k)$-tensor $A$ on V, define a symmetric (resp. skew-symmetric) $(0,k)$-tensor $\mathrm{Sym}(A)$ (resp. $\mathrm{Alt}(A)$) by

$$\mathrm{Sym}(A)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} A(v_{\sigma(1)},\dots,v_{\sigma(k)}),$$

$$\text{resp. } \mathrm{Alt}(A)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} (-1)^{\mathrm{sgn}(\sigma)} A(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

- Think of $\mathrm{Sym}$ and $\mathrm{Alt}$ as linear maps from $\mathrm{T}^0_k(\mathsf{V})$ to $\mathrm{T}^0_k(\mathsf{V})$.

- A $(0,3)$-tensor $A$ is ***gyroscopic*** if $A(u,v,w) = -A(v,u,w)$
  is ***torsional*** if $A(u,v,w) = -A(u,w,v)$
  $\mathrm{Gyr}(\mathsf{V})$: gyroscopic tensors
  $\mathrm{Tor}(\mathsf{V})$: torsional tensors
- For a Riemannian metric $\mathbb{G}$, define $\mathrm{KE}_{\mathbb{G}}\colon \mathsf{TQ} \to \mathbb{R}$ by $\mathrm{KE}_{\mathbb{G}}(v_q) = \frac{1}{2}\mathbb{G}(v_q, v_q)$.
- An affine connection $\nabla$ is ***$\mathbb{G}$-energy-preserving*** if $\mathscr{L}_{\gamma''(t)}\mathrm{KE}_{\mathbb{G}}(\gamma'(t)) = 0$ for every geodesic $\gamma$ of $\nabla$.

**Proposition 5** *The following are equivalent:*

*(i)* $\nabla$ *is $\mathbb{G}$-energy preserving;*

*(ii)* $\nabla\mathbb{G} \in \Gamma^\infty(\ker(\mathrm{Sym}))$*;*

*(iii)* $D_{\nabla,\mathbb{G}} \in \Gamma^\infty(\ker(\mathrm{Sym}))$*;*

*(iv)* *there exists tensor fields* $\Omega_{\nabla,\mathbb{G}} \in \Gamma^\infty(\mathrm{T}\bigwedge^3(\mathsf{TQ}))$*,* $B_{\nabla,\mathbb{G}} \in \Gamma^\infty((\mathrm{Gyr}(\mathsf{TQ}) \cap \ker(\mathrm{Alt})))$*, and* $\hat{T}_{\nabla,\mathbb{G}} \in \Gamma^\infty((\mathrm{Tor}(\mathsf{TQ}) \cap \ker(\mathrm{Alt}))$ *such that*

$$\mathbb{G}(\nabla_X Y, Z) = \mathbb{G}(\overset{\mathbb{G}}{\nabla}_X Y, Z) + B_{\nabla,\mathbb{G}}(Z,X,Y) + \hat{T}_{\nabla,\mathbb{G}}(Z,X,Y) + \Omega_{\nabla,\mathbb{G}}(Z,X,Y),$$

*for all* $X, Y, Z \in \Gamma^\infty(\mathsf{TQ})$*.*

- *Discussion:*
  1. If $B$ is a gyroscopic tensor field, then there exists a unique energy-preserving, torsion-free affine connection $\nabla$ such that

     $$\mathbb{G}(\nabla_X X, Y) = \mathbb{G}(\overset{\mathbb{G}}{\nabla}_X X, Y) + B(Y, X, X)$$

     for all $X, Y \in \Gamma^\infty(\mathsf{TQ})$.
     Explicitly, $\nabla$ is defined by

     $$\mathbb{G}(\nabla_X Y, Z) = \mathbb{G}(\overset{\mathbb{G}}{\nabla}_X Y, Z) + B_{\nabla,\mathbb{G}}(Z, X, Y),$$

     where $B_{\nabla,\mathbb{G}} = B - \mathrm{Alt}(B)$.
  2. Changes the kinetic energy/quadratic gyroscopic force determination into a purely affine connection problem:
     Find a Riemannian metric $\mathbb{G}_{\mathsf{cl}}$ and a $\mathbb{G}_{\mathsf{cl}}$-energy preserving connection $\overset{\mathsf{cl}}{\nabla}$ such that
     $\overset{\mathsf{cl}}{\nabla}_{\gamma'(t)}\gamma'(t) - \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t) \in \mathbb{G}_{\mathsf{ol}}^\sharp(\mathcal{F}_{\gamma(t)})$.

## Setting up the kinetic shaping problem for integrability tests[1]

- Define

  $$\mathrm{ES}(\mathsf{Q}) = \Sigma_2^+(\mathsf{TQ}) \times (\mathrm{Gyr}(\mathsf{TQ}) \cap \ker(\mathrm{Alt}))$$

  and

  $$P_{\mathsf{KS}}(\mathbb{G}_{\mathsf{ol}})_q = \big\{(j^1\mathbb{G}(q), j^1B(q)) \in J^1(\mathrm{ES}(\mathsf{Q}))\big| \\ (\mathrm{LC}(j^1\mathbb{G}(q)) - \mathrm{LC}(j^1\mathbb{G}_{\mathsf{ol}}(q)) + \mathbb{G}^\sharp B \in \mathbb{G}_{\mathsf{ol}}^\sharp(\mathcal{F} \otimes \mathrm{TS}^2(\mathsf{TQ}))\big\}.$$

**Proposition 6** *A Riemannian metric $\mathbb{G}_{\mathsf{cl}}$ and a gyroscopic tensor field $B_{\mathsf{cl}}$ solve the kinetic energy/quadratic gyroscopic problem if and only if the 1-jet of the section $q \mapsto (\mathbb{G}_{\mathsf{cl}}(q), B_{\mathsf{cl}}(q))$ takes values in $P_{\mathsf{KS}}(\mathbb{G}_{\mathsf{ol}})$.*

- This result puts the kinetic shaping problem in a form where the techniques of Spencer, Serre, Quillen, Goldschmidt, etc. are applicable.

[1] Ibid

# Energy shaping and linearisation

## Problem statement

- Given: A simple mechanical control system $\Sigma_{\mathsf{nonlin}} = (\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \mathscr{F})$ with $q_0$ an equilibrium point and $\mathcal{F}$ regular.
- Let $\Sigma_{\mathsf{lin}} = (\mathsf{T}_{q_0}\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}(q_0), \operatorname{Hess} V_{\mathsf{ol}}(q_0), F)$ be its linearisation at $q_0$.

**Problem 1** When can a linear energy shaping feedback for the linearisation be implemented on the full system, and the implementation is ensured to also be energy shaping? •

- What does "implemented" mean?
- If one is working in a particular set of coordinates, one "implements" without thinking about it.

- A coordinate-free version is the following.

**Definition 7** A ***near identity diffeomorphism*** at $q_0 \in \mathsf{Q}$ is a triple $(\chi, \mathcal{U}_0, \mathcal{U}_1)$, where

(i) $\mathcal{U}_0 \subset \mathsf{T}_{q_0}\mathsf{Q}$ is a neighborhood of $0_{q_0}$,

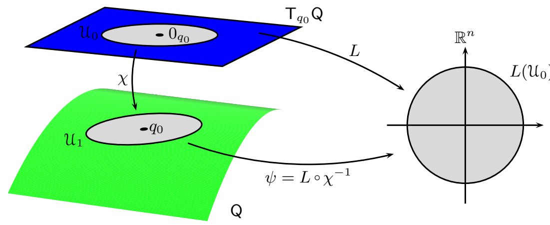
(ii) $\mathcal{U}_1 \subset \mathsf{Q}$ is a neighborhood of $q_0$, and

(iii) $\chi\colon \mathcal{U}_0 \to \mathcal{U}_1$ is a diffeomorphism satisfying

  (a) $\chi(0_{q_0}) = q_0$ and

  (b) $T_{0_{q_0}}\chi = \mathrm{id}_{\mathsf{T}_{q_0}\mathsf{Q}}$ (where we make the natural identification of $\mathsf{T}_{0_{q_0}}(\mathsf{T}_{q_0}\mathsf{Q})$ with $\mathsf{T}_{q_0}\mathsf{Q}$). •

- Near identity diffeomorphisms have a simple relationship with coordinate charts:




- An ***implementation*** of a linear feedback $u_{\mathsf{lin}} \colon \mathsf{T}_{q_0}\mathsf{Q} \oplus \mathsf{T}_{q_0}\mathsf{Q} \to \mathbb{R}^m$ using a near identity diffeomorphism $(\chi, \mathcal{U}_0, \mathcal{U}_1)$ is the control law $u_{\mathsf{nonlin}} = u_{\mathsf{lin}} \circ T\chi^{-1}$.

## Linearisation and potential shaping

**Proposition 7** *For $(\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \mathscr{F})$ the following are equivalent:*

*(i) $\mathcal{F}$ is integrable;*

*(ii) there exists a family $\tilde{\mathscr{F}}$ of input one-forms[1] and a near identity diffeomorphism such that every closed-loop potential of $\Sigma_{\mathsf{lin}}$ can be implemented as a potential shaping feedback for $\Sigma_{\mathsf{nonlin}}$.*

- *Punchline:* The obstructions to implementing a linearly shaped potential on the nonlinear system are the same as the obstructions to nonlinear potential shaping.

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[1] Equivalent to $\mathscr{F}$ in the sense that $\tilde{\mathcal{F}} = \mathcal{F}$.

## Linearisation and dissipative and gyroscopic forces

**Proposition 8** *For* $(\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \mathscr{F})$ *the following are equivalent:*

*(i)* $\mathcal{F}$ *is integrable;*

*(ii) there exists a family* $\tilde{F}$ *of input one-forms and a near identity diffeomorphism such that every closed-loop dissipative (resp. gyroscopic) force for* $\Sigma_{\mathsf{lin}}$ *can be implemented as a closed-loop dissipative (resp. gyroscopic) force for* $\Sigma_{\mathsf{nonlin}}$*.*

- *Punchline:* The obstructions implementing linear dissipative and gyroscopic forces on the nonlinear system are the same as the obstructions to nonlinear potential shaping.

## Interesting open question

**Question 1** When can a kinetic/potential shaping feedback for $\Sigma_{\mathsf{lin}}$ be implemented as a kinetic/potential shaping feedback for $\Sigma_{\mathsf{nonlin}}$?

## Open problems

1. Integrability of potential shaping p.d.e.
2. Integrability of kinetic shaping p.d.e.
3. Computable necessary or sufficient conditions for integrability.
4. What closed-loop kinetic energies allow useful potential shaping?
5. Complete the linearisation picture.
6. Closed-loop stability considerations.