What is control theory?

- Control theory is the study of the manipulation of dynamical processes to achieve desired objectives.

- There are many sorts of models considered in control theory. Let us fix one:

\[ \dot{x}(t) = f(x(t), u(t)), \]
\[ y(t) = g(x(t), u(t)). \]

1. The variable \( x \in \mathbb{R}^n \) is the state whose behaviour is being controlled.
2. The variable \( u \in \mathbb{R}^m \) is the control or input which we can specify as we like.
3. The variable \( y \in \mathbb{R}^p \) is the output, and might represent what can be measured.
Simple examples

• Servo-motor:

1. State: Angular position and velocity of flywheel (and possibly armature current in motor, for accurate model).
2. Input: Voltage $V(t)$.
4. Typical control objective: Design $V(t)$ so that a desired flywheel angular velocity $\omega(t)$ is achieved and maintained.

• Coupled tanks:

1. States: Heights $h_1(t)$ and $h_2(t)$ of water in tanks.
2. Input: Input flow $F_{in}(t)$.
3. Output: One or both tank heights.
4. Typical control objective: Design $F_{in}(t)$ so that desired heights $h_1(t)$ and $h_2(t)$ are achieved and maintained.
Typical control problems

1. **Stabilisation**: Design the control $u_{\text{des}}$ (as a function of state, or time, or possibly both) so that a desired state $x_0$ is a stable equilibrium for the differential equation

$$\dot{x}(t) = f(x(t), u_{\text{des}}(x,t)),$$

or $u_{\text{des}}(t)$ or $u_{\text{des}}(x)$.

2. **Output tracking**: Obtain $u_{\text{des}}$ so that the output follows a specified trajectory $y_{\text{des}}(t)$.

3. **Motion planning**: Steer the state from $x_1$ to $x_2$.

4. **Optimal control**: Do any of the above while minimising some cost function, e.g., time, or "control energy."

Typical approaches

- **Linear systems**:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

1. Much—a huge amount—has been done using tools from linear algebra and functional analysis.
2. This is what is typically used in practice.

- **Nonlinear systems**:

1. Linearise—this works sometimes.
2. In general, nonlinear control theory is very difficult.
3. Need to focus on structure to gain understanding.
Geometric control theory

• Let us specialise a little to systems that are control-affine:

\[ \dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u_a(t)f_a(x(t)), \]  

(1)

where \( f_0 \) is the drift vector field and \( f_1, \ldots, f_m \) are the control vector fields.

• Forget about outputs to keep things simple.

• In geometric control theory we are interested in understanding the properties of system (1) from the point of view of differential geometry.

• I will not assume you know any differential geometry...
• **Question:** Why is the reachable set important?

• **Answer:**
  1. It gives some idea of what is possible as far as control objectives. For example, maybe the reachable set can tell you that it is not possible to steer between states $x_1$ and $x_2$.
  2. Properties of the reachable set appear (although often are hidden) as hypotheses in many design procedures.
  3. There are non-obvious and important connections between the reachable set and optimal control theory.
  4. There should be (as yet unexplored) relationships between the reachable set and the stabilisation problem.

**Exploring the reachable set**

• The definition of the reachable set is not useful, because to define it requires computing solutions to differential equations—this is impossible in general.

• **Question:** Are there computable ways of characterising the reachable set?

• Consider the following simple control system:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x).$$

• Apply the control

$$u(t) = \begin{cases} 
(1, 0), & 0 \leq t < T, \\
(0, 1), & T \leq t < 2T, \\
(-1, 0), & 2T \leq t < 3T, \\
(0, -1), & 3T \leq t \leq 4T.
\end{cases}$$
• Where does $x(4T)$ end up? (Note that $x(4T)$ is clearly in the reachable set.)

• We determine (e.g., by Taylor expansion) that

$$x(4T) = x(0) + T^2 [f_1, f_2](x(0)) + \ldots, \quad [f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2.$$ 

• $[f_1, f_2]$ is the Lie bracket of $f_1$ and $f_2$.

• By applying suitable controls to our general system, one may move in the directions

$$f_0, f_1, \ldots, f_m,$$

$$[f_a, f_b], \quad a, b = 0, \ldots, m,$$

$$[f_a, [f_b, f_c]], \quad a, b, c = 0, \ldots, m,$$

etc.

A mechanical exhibition of the Lie bracket

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A theorem on the nature of the reachable set

Theorem 1

Procedure: Compute Lie brackets

\[ [f_{a_1}, [f_{a_2}, \ldots, [f_{a_{k-1}}, f_{a_k}]]], \quad k \in \mathbb{Z}^+, \quad a_1, \ldots, a_k \in \{0, 1, \ldots, m\}. \]

Check: At some point in the computation, do some collection of these Lie brackets evaluated at \( x_0 \) form a basis?

Conclusion: If so, \( \text{int}(\mathcal{R}(x_0, \leq T)) \neq \emptyset \).

Is the Lie bracket important?

- Our previous analysis and theorem suggest that the Lie bracket is interesting in geometric control theory.
- It is also important in differential geometry, physics, some areas of partial differential equations.
- It is also an example of a general structure in algebra known as a Lie algebra. Associated with these are Lie groups.
- Thus Lie brackets appear in various contexts in mathematics, as well as being essential in geometric control theory.
Refining the reachable set

\[ \text{int}(\mathcal{R}(x_0, \leq T)) \neq \emptyset \]

\[ x_0 \in \text{int}(\mathcal{R}(x_0, \leq T)) \]

Accessibility

Controllability

- Accessibility is essentially exactly characterised by the previous theorem.
- Controllability is “impossible” (precisely, it is NP-hard, in the language of computational complexity).

Mechanical systems
Geometric mechanics

• Apart from control theory, mechanics has its own very interesting mathematical structure.

• The most important part of the physical model is the kinetic energy leads to an interesting geometric structure called a Riemannian metric.

• Also interesting are nonholonomic constraints (as in last two of the examples).

• The structure associated with a nonholonomic constraint is a distribution relationship between Riemannian metric and constraint distribution gives lots of interesting problems.

A simple example in detail

• Hovercraft system:
  1. Question: Is the system accessible?
  2. Answer: Yes (easy).
  3. Question: Is the system controllable?
  4. Answer: Yes (a little harder).
  5. Question: Can we design an algorithm to steer from state to state?
  6. Answer: Yes, if we are quite clever.
  7. Question: Can we design an algorithm to stabilise a desired state?
  8. Answer: Yes, but we do not understand this very well.
Make the example harder

1. Question: Is the system accessible?
2. Answer: Yes (easy).
3. Question: Is the system controllable?
4. Answer: No, at least not locally (nontrivial).
5. Question: Can we design an algorithm to steer from state to state?
7. Question: Can we design an algorithm to stabilise a desired state?
8. Answer: Unknown

Make the example different

1. Question: Is the system accessible?
2. Answer: Yes (easy).
3. Question: Is the system controllable?
4. Answer: No, at least not locally (getting really difficult now).
5. Question: Can we design an algorithm to steer from state to state?
7. Question: Can we design an algorithm to stabilise a desired state?
8. Answer: Unknown
**Punchline**  Even easy problems can be very difficult.

**Motion planning for the easy planar body**

- Movies for the planar body.
Nonholonomic mechanics: snakeboard example

• Snakeboard corporate movies.
• Snakeboard gaits.

Snakeboard motion planning

• The movies suggest that the snakeboard is controllable. It is.

• Problem: Can one design an algorithm to steer the snakeboard from a desired initial position to a desired final position?

• Answer: Yes!
But where’s the mathematics?

- Control theory is widely practised as an engineering discipline.
- But it is also a mathematical subject in its own right.
- It has many branches.
- Linear control theory:
  1. linear differential equations;
  2. linear algebra;
  3. complex function theory;
  4. measure theory;
  5. functional analysis;
  6. operator theory;
  7. convex analysis.

- Nonlinear control theory:
  1. linear algebra;
  2. advanced differential equations;
  3. measure theory;
  4. “simple” differential geometry.

- Geometric control theory:
  1. linear algebra;
  2. advanced differential equations;
  3. measure theory;
  4. differential geometry;
  5. theory of distributions;
  6. analytic differential geometry (e.g., no partitions of unity);
• Control theory for mechanical systems:
  1. all the stuff from geometric control theory plus
  2. Riemannian geometry;
  3. affine differential geometry;
  4. Lie groups;
  5. some physics, if it interests you.

• **Important fact:** Each branch of control theory also has its own unique mathematical problems. That is, control theory is a subject in mathematics.

**Summary**

• Control theory is a broad subject, which uses a huge variety of mathematics, and possesses its own intricate mathematical problems.

• Mechanical control systems provide a class of systems with rich geometric structure.

• There is much work to be done here.
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