Control theory without controls

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What are we trying to do?

- In control theory, a commonly encountered class of systems is represented by equations like

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t))$$

  for vector fields $\mathscr{F} = \{f_0, f_1, \dots, f_m\}$ on a manifold M. The control $t \mapsto u(t)$ takes values in $U \subset \mathbb{R}^m$.

- The geometrically interesting thing here is the affine distribution $\mathcal{A}_{\mathscr{F}}$ defined by

$$\mathcal{A}_{\mathscr{F},x} = \left\{ f_0(x) + \sum_{a=1}^{m} u^a f_a(x) \;\middle|\; u \in \mathbb{R}^m \right\}.$$

  Objective: Frame control theory in terms of the affine distribution $\mathcal{A}_{\mathscr{F}}$ rather than the specific choice of generators $\mathscr{F}$.

- In particular, we are interested in the geometry of controllability, stabilisability, and relationships between controllability and stabilisability.
- This seems obvious. Why is there any "content" here?
  Answer: Because the control set U actually matters, but in subtle and nonobvious ways.
  ⟶ Need to really understand some control theory.
  ⟶ Learn things about affine distributions using control theory.
- This is completely vague, so let's be precise.

~~Control~~ Affine systems

Definitions (and lots of them)

Definition 1 An ***affine distribution*** on M is a subset $\mathcal{A} \subset \mathsf{T}\mathsf{M}$ such that, for $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{N}$ of x_0 and vector fields $X_0, X_1, \dots, X_k$ on $\mathcal{N}$ such that

$$\mathcal{A}_x \triangleq \mathcal{A} \cap \mathsf{T}_x\mathsf{M} = \left\{X_0(x) + \textstyle\sum_{j=1}^{k} u^j X_j(x) \;\middle|\; u \in \mathbb{R}^k\right\}$$

for each $x \in \mathcal{N}$. •

Definition 2 An ***affine system*** $\mathscr{A}$ in an affine distribution $\mathcal{A}$ is an assignment of a subset $\mathscr{A}(x) \subset \mathcal{A}_x$ for each $x \in \mathsf{M}$ such that

(i) $\operatorname{aff}(\mathscr{A}(x)) = \mathcal{A}_x$ and

(ii) some regularity conditions are satisfied (e.g., continuity with respect to the Hausdorff metric in coordinates). •

Definition 3 An affine system $\mathscr{A}$ in an affine distribution $\mathcal{A}$ is ***proper*** at $x_0 \in \mathsf{M}$ if $0_{x_0} \in \mathrm{int}_{\mathrm{aff}(\mathscr{A}(x_0))}(\mathrm{conv}(\mathscr{A}(x_0)))$. •

- *Intuition:* Proper $\longrightarrow$ can push in all possible directions with controls (assumes that $L(\mathcal{A})_{x_0} = \mathcal{A}_{x_0}$).

Definition 4 A ***trajectory*** for an affine system $\mathscr{A}$ in an affine distribution $\mathcal{A}$ is a locally absolutely continuous curve $\xi \colon I \to \mathsf{M}$ such that $\xi'(t) \in \mathscr{A}(\xi(t))$ for a.e. $t \in I \subset \mathbb{R}$. •

- *Punchline:* We have trajectories but no controls.

Controllability definitions

- Usual reachable set definitions:

$$\mathcal{R}_{\mathscr{A}}(x_0, T) = \{\xi(T) \mid \ \xi \colon [0,T] \to \mathsf{M} \text{ is a trajectory for } \mathscr{A} \text{ such that } \xi(0) = x_0\},$$
$$\mathcal{R}_{\mathscr{A}}(x_0, \leq T) = \cup_{t\in[0,T]} \mathcal{R}_{\mathscr{A}}(x_0, t).$$

- Predictable controllability definition:

Definition 5 An affine system $\mathscr{A}$ is ***small-time locally controllable*** (***STLC***) from $x_0 \in \mathsf{M}$ if there exists $T > 0$ such that $\mathrm{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq t)) \neq \varnothing$ for all $t \in [0, T]$. •

- One also has the more basic notion of accessibility, but I will suppose this is well understood (and it is, even in the affine distribution world).

  ⟶ We will always assume we are dealing with accessible systems.

- The preceding definitions are the exact analogue of the usual controllability definitions; if we stop here, we have done nothing new.

- That what is missing is the exact rôle of the affine distribution $\mathcal{A}$: the definitions involve $\mathscr{A}$ is an essential way.

- We get around this with the following (important, I claim) definitions.

Definition 6 An affine distribution $\mathcal{A}$ is:

(i) ***properly small-time locally controllable*** (***PSTLC***) from x_0 if $\mathscr{A}$ is STLC from x_0 for any affine system $\mathscr{A}$ in $\mathcal{A}$ that is proper at x_0;

- (*Intuition:* PSTLC ⟶ If it is not "inconceivable" that an affine system $\mathscr{A}$ in $\mathcal{A}$ be STLC, then it is STLC.)

(ii) ***small-time locally uncontrollable*** (***STLUC***) from x_0 if $\mathscr{A}$ is not STLC from x_0 for any affine system $\mathscr{A}$ in $\mathcal{A}$ for which $\mathscr{A}(x_0)$ is compact;

- (*Intuition:* STLUC ⟶ Any "feasible" affine system $\mathscr{A}$ in $\mathcal{A}$ is not STLC.)

(iii) ***conditionally small-time locally controllable*** (***CSTLC***) from x_0 if it is not PSTLC from x_0 but there exists some affine system $\mathscr{A}$ in $\mathcal{A}$ such that $\mathscr{A}(x_0)$ is compact and $\mathscr{A}$ is STLC from x_0. •

- (*Intuition:* CSTLC ⟶ There is *some* "feasible" affine system $\mathscr{A}$ in $\mathcal{A}$ that is STLC.)

Examples 7 (All using standard language)

1. A system with controllable linearisation at x_0 is PSTLC from x_0. (We will say shortly what "controllable linearisation" means in terms of affine distributions.)
2. The system

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_1^2$$

is STLUC from $(0, 0)$.

3. For a control-affine system $\mathscr{F} = \{f_0, f_1, \dots, f_m\}$ let $\mathscr{F}_1 = \{f_1, \dots, f_m\}$ and $\mathcal{F}_{1,x} = \mathrm{span}_{\mathbb{R}}(f_1(x), \dots, f_m(x))$. Then:
   (a) $f_0(x_0) \notin \mathcal{F}_{1,x_0}$ ⟶ STLUC from x_0;
   (b) $f_0(x_0) = 0_{x_0}$, $f_0 \notin \mathrm{span}_{C^\infty(\mathsf{M})}(\mathscr{F}_1)$, x_0 a regular point for $\mathcal{F}_1$, and $\mathrm{Lie}^{(\infty)}(\mathcal{F}_1)_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$ ⟶ CSTLC from x_0.[1]

[1] e.g., Bianchini/Stefani, *SIAM J. Control Optim.*, **31**(4), 900-917, 1993.

Stabilisation definitions

Definition 8 A ***state feedback*** for an affine system $\mathscr{A}$ in an affine distribution $\mathcal{A}$ is a vector field X on M such that $X(x) \in \mathscr{A}(x)$. •

- The degree of differentiability of state feedback is crucial in any discussion of stabilisability. One will generally wish to allow discontinuous state feedback. Let us first consider state feedback that is at least continuous.

Definition 9 Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \cup \{\omega\}$. An affine system $\mathscr{A}$ in $\mathcal{A}$ is ***C^r-locally asymptotically stabilisable*** (**LAS^r**) to $x_0 \in \mathsf{M}$ if there exists a neighbourhood $\mathcal{N}$ of x_0 and a C^r-state feedback X such that x_0 is a locally asymptotically stable equilibrium point for $X|\mathcal{N}$. •

- First of all, these notions should probably be modified with various additional adjectives like "almost," meaning that the feedback is C^r on a punctured neighbourhood of x_0.
- We are now confronted with the same problem as we were in our presentation of controllability: our definition of stabilisation involves the affine system $\mathscr{A}$ and not just the affine distribution $\mathcal{A}$.
- One overcomes this in the same way as for controllability: by introducing the notions PLAS^r, LAUS^r, and CLAS^r for $\mathcal{A}$.

Example 10 Here is a system that is CLAS$^\omega$. Take $\mathsf{M} = \mathbb{R}$ and $\mathcal{A}_x$ the affine distribution (distribution, actually) generated by $x\frac{\partial}{\partial x}$. The affine system

$$\mathscr{A}_1(x) = \left\{(1+u)x\tfrac{\partial}{\partial x} \;\middle|\; u \in [-\tfrac{1}{2}, \tfrac{1}{2}]\right\}$$

is proper at 0 but not LAS^0 (or even open-loop stabilisable) to 0 but the affine system

$$\mathscr{A}_2(x) = \left\{(1+u)x\tfrac{\partial}{\partial x} \;\middle|\; u \in [-2, 2]\right\}$$

is compact at 0 and is LAS$^\omega$ to 0.

(In case it is not obvious, this is a bilinear system.) •

- Note that the affine distribution is singular at 0: singularities cause some odd phenomenon.

Discontinuous stabilisation

- It is known that open-loop stabilisation (i.e., the weakest possible form) is equivalent to stabilisation using state feedback, provided that the state feedback is allowed to be suitably discontinuous.[1,2]
- We need the appropriate notion of open-loop stabilisation to capture this.

Definition 11 An affine system $\mathscr{A}$ in $\mathcal{A}$ is ***locally asymptotically controllable*** (***LAC***) to x_0 if there exists a neighbourhood $\mathcal{N}$ of x_0 such that, for each $x \in \mathcal{N}$, there exists a trajectory $\xi\colon [0, \infty[\to \mathcal{N}$ for $\mathscr{A}$ such that $\xi(0) = x$ and $\lim_{t\to\infty} \xi(t) = x_0$.

- Now define the notions PLAC, LAUC, and CLAC for $\mathcal{A}$.

[1] Clarke, Ledyaev, Sontag, and Subotin, *IEEE Trans. Automat. Control*, **42**(10), 1394–1407, 1997

[2] Ancona and Bressan, *ESAIM Control Optim. Calc. Var.*, **4**, 445–471, 1999

Results

- There are almost no results out there characterising controllability, stabilisability, and asymptotic controllability in terms of affine distributions.
  ⟶ There is a huge amount of work to be done here.

Some (trivial) controllability results

Theorem 12 (Zeroth-order controllability)

(i) If $\mathcal{A}_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$ then $\mathcal{A}$ is PSTLC from x_0.

(ii) If $0_{x_0} \notin \mathcal{A}_{x_0}$ then $\mathcal{A}$ is STLUC from x_0.

- To state the linearisation controllability result:
  If V is a $\mathbb{R}$-vector space, if $S \subset$ V, and if $\mathscr{L} \subset$ End(V), denote by $\langle \mathscr{L}, S \rangle$ the smallest subspace containing S which is invariant under each $L \in \mathscr{L}$.
- Let $Z_{x_0}(\mathcal{A}) = \{X \in \Gamma(\mathcal{A}) \mid X(x_0) = 0_{x_0}\}$.
- Note that $Z_{x_0}(\mathcal{A})$ can be thought of as a subset of End(T_{x_0}M) by $X \mapsto L_X$ with $L_X(v) = [V, X](x_0)$ where V is any vector field extending $v \in \mathsf{T}_{x_0}$M.

Theorem 13 (Linearised controllability) *If*

(i) $0_{x_0} \in \mathcal{A}_{x_0}$ *and*

(ii) $\langle Z_{x_0}(\mathcal{A}), \mathcal{A}_{x_0} \rangle = \mathsf{T}_{x_0}$M

then $\mathcal{A}$ *is PSTLC from* x_0*.*

A simple but illustrative example relating controllability and stabilisation

- Let $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and consider

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{u}$$
$$\dot{\boldsymbol{x}}_2 = \boldsymbol{Q}(\boldsymbol{x}_1)$$

  where $\boldsymbol{Q}$ is a quadratic function.
- Example is "simplest" one that is interesting, e.g., there are potential obstructions to controllability.
- *Controllability from* $\mathbf{0}_n$*:*
  1. PSTLC if $\mathbf{0}_{n-m} \in \mathrm{int}(\mathrm{conv}(\mathrm{image}(\boldsymbol{Q})))$;
  2. STLUC otherwise.
- We have a pair (one for PSTLC and one for STLUC) of very general second-order theorems; it is not possible to say anything about these here due to lack of time.

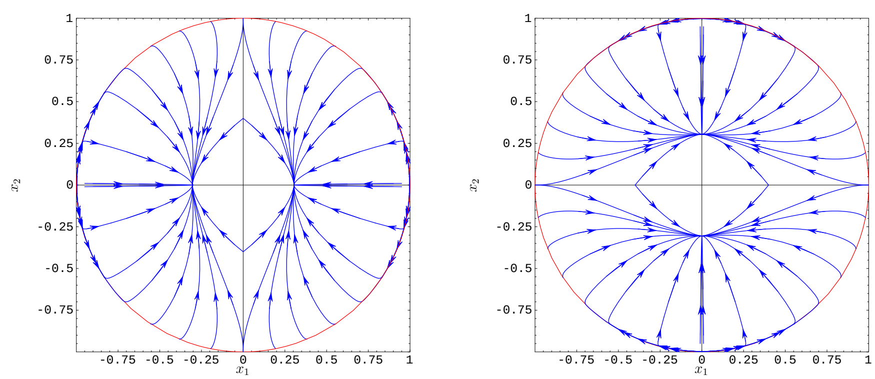
- *Stabilisability to $\mathbf{0}_n$:*
  1. Fact: PSTLC from $\mathbf{0}_n$ if and only if PLAC to $\mathbf{0}_n$.
  2. Conjecture: not stabilisable using continuous feedback. Brockett's topological necessary condition[1] for continuous stabilisability does not always give a result for these systems. Apply Coron's condition?[2]
  3. Objective: Design stabilising state feedback. We have done this only for the simplest possible case:

     $$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1^2 - x_2^2.$$

     Even in this case it is really nontrivial to design the state feedback. But...the geometry is helpful.

[1] *Differential Geometric Control Theory*, 181–191, Birkhäuser, 1983
[2] *Systems Control Lett.*, **14**(3), 227–232, 1990

- The stabilising feedback boils down to the following picture that should be thought of as giving a system on a sphere (glue the disks on their boundary):



- One seeks a homogeneous control and reduces to the homogeneous sphere.

Conclusion

A truly geometric theory of affine systems
is a long way from being developed.