Potential energy shaping after kinetic energy shaping

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1. The problem

- The problem is simple:

  Problem 1 Given an open-loop mechanical system with kinetic energy $\mathbb{G}_{\text{ol}}$, potential energy V_{ol}, and control forces $F^1, \dots, F^m$, design controls such that

  (i) the closed-loop system is a mechanical system with kinetic energy $\mathbb{G}_{\text{cl}}$, potential energy V_{cl}, and possibly some other closed-loop external forces (e.g., gyroscopic forces) and such that

  (ii) a desired equilibrium point q_0 is stable.

2. The problem, redux

2.1. A breakdown of energy shaping feedback

- Here is a possible design procedure.

Procedure 2 Given: An open-loop system $(\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \{F^1, \dots, F^m\})$:

(i) find a control u_{kin} that shapes the kinetic energy to $\mathbb{G}_{\mathsf{cl}}$, possibly by also adding some external closed-loop forces;

(ii) find a control u_{pot} that, with the given closed-loop kinetic energy $\mathbb{G}_{\mathsf{cl}}$, gives a closed-loop potential V_{cl};

(iii) take $u = u_{\mathsf{kin}} + u_{\mathsf{pot}}$.

2.2. What constitutes a solution to the energy shaping control problem?

- If all one is interested in is stabilisation of a point q_0, then a solution is any energy shaping feedback that does this.
- But let's be more ambitious than this...
- The key element in determining the character of the closed-loop system is the closed-loop potential. For example, stability of q_0 means $\boldsymbol{d}V_{\mathsf{cl}}(q_0) = 0$ and $\operatorname{Hess} V(q_0) > 0$.
- So...

Problem 3 Describe all possible closed-loop potential functions.

- This has two components:
  1. Describe all possible closed-loop kinetic energies.
  2. For a given closed-loop kinetic energy, describe the possible closed-loop potential energies.
- We are interested here in the second component.

3. The result

3.1. The setup

- We have an open-loop system $(\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \{F^1, \ldots, F^m\})$ and have applied a control to shape the kinetic energy to $\mathbb{G}_{\mathsf{cl}}$.
- Define $\Lambda_{\mathsf{cl}} = \mathbb{G}_{\mathsf{ol}}^{\flat} \circ \mathbb{G}_{\mathsf{cl}}^{\sharp}$.
- Let $\mathcal{F}$ be the codistribution generated by $\{F^1, \ldots, F^m\}$ (control forces are thus $\mathcal{F}$-valued). Assume constant rank.
- Let $\mathcal{F}_{\mathsf{cl}}$ be the codistribution $\Lambda_{\mathsf{cl}}^{-1}(\mathcal{F})$.

Lemma 4 *Given:* $\mathbb{G}_{\mathsf{ol}}$, V_{ol}, *and* $\mathbb{G}_{\mathsf{cl}}$.

A force F taking values in $\mathcal{F}$ gives a closed-loop potential V_{cl} if and only if

$$F(q) = \Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V_{\mathsf{cl}}(q) - \boldsymbol{d}V_{\mathsf{ol}}(q), \qquad q \in \mathsf{Q}.$$

3.2. The potential energy shaping partial differential equation

- Define $\mathsf{Q}_{\mathbb{R}} = \mathsf{Q} \times \mathbb{R}$ and $\pi\colon \mathsf{Q}_{\mathbb{R}} \to \mathsf{Q}$ by $\pi(q, V) = q$.
- A potential function V defines a section of $\mathsf{Q}_{\mathbb{R}}$: $q \mapsto (q, V(q))$.
- Let $\mathscr{D}_{\boldsymbol{d}}$ be the first-order differential operator on $\mathsf{Q}_{\mathbb{R}}$ given by $\mathscr{D}_{\boldsymbol{d}}(V) = \boldsymbol{d}V$. This gives a map $\Phi_{\boldsymbol{d}}\colon \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \to \mathsf{T}^*\mathsf{Q}$ satisfying $\Phi_{\boldsymbol{d}}(j_1V(q)) = \boldsymbol{d}V(q)$.
- Abbreviate $\alpha_{\mathsf{cl}} = \Lambda_{\mathsf{cl}}^{-1} \circ \boldsymbol{d}V_{\mathsf{ol}}$.
- Let $\pi_{\mathcal{F}_{\mathsf{cl}}}\colon \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}/\mathcal{F}_{\mathsf{cl}}$ be the canonical projection.
- Define

$$\mathsf{R}_{\mathsf{pot}} = \{p \in \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \mid \pi_{\mathcal{F}_{\mathsf{cl}}} \circ \Phi_{\boldsymbol{d}}(p) = \pi_{\mathcal{F}_{\mathsf{cl}}} \circ \alpha_{\mathsf{cl}}(q),\ \pi_1(p) = q\},$$

  where $\pi_1\colon \mathsf{J}_1\mathsf{Q}_{\mathbb{R}} \to \mathsf{Q}$ is the canonical projection.

Proposition 5 *A section F of $\mathcal{F}$ is a potential energy shaping feedback if and only if $F = \Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V - \boldsymbol{d}V_{\mathsf{ol}}$ for a solution V to $\mathsf{R}_{\mathsf{pot}}$.*

- What's the point of all the fanciness?
- You get a partial differential equation in the framework for applying the Goldschmidt[1] theory for integrability of partial differential equations.

[1] *J. Differential Geom.*, **1**, 269–307, 1967

3.3. The statement

- Let $\mathsf{I}_2(\mathcal{F}_{\mathsf{cl}})$ be the two-forms in the algebraic ideal generated by $\mathcal{F}_{\mathsf{cl}}$.

Theorem 6 *Let $(\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \mathscr{F})$ be an analytic simple mechanical control system and let $\mathbb{G}_{\mathsf{cl}}$ be an analytic Riemannian metric. Let $p_0 \in \mathsf{R}_{\mathsf{pot}}$ and let $q_0 = \pi_1(p_0)$. Assume that q_0 is a regular point for $\mathcal{F}$ and that $\mathcal{F}_{\mathsf{cl}}$ is integrable in a neighbourhood of q_0. Then the following statements are equivalent:*

(i) there exists a neighbourhood $\mathcal{U}$ of q_0 and an analytic potential energy shaping feedback F defined on $\mathcal{U}$ which satisfies $\Phi_{\boldsymbol{d}}(p_0) = F_{\mathsf{cl}}(q_0) + \alpha_{\mathsf{cl}}(q_0)$;

(ii) there exists a neighbourhood $\mathcal{U}$ of q_0 such that $\boldsymbol{d}\alpha_{\mathsf{cl}}(q) \in \mathsf{I}(\mathcal{F}_{\mathsf{cl},q})$ for each $q \in \mathcal{U}$.

Moreover, if $V_{\mathsf{cl},1}$ and $V_{\mathsf{cl},2}$ are two closed-loop potential functions, then $\boldsymbol{d}(V_{\mathsf{cl},1} - V_{\mathsf{cl},2})(q) \in \mathcal{F}_{\mathsf{cl},q}$ for each $q \in \mathsf{Q}$.

3.4. The working version

- If $\mathcal{F}_{\mathsf{cl}}$ is not integrable, replace it with the largest integrable codistribution contained in it.
- Since $\mathcal{F}_{\mathsf{cl}}$ is integrable choose coordinates $(q^1, \dots, q^n)$ for Q such that

$$\mathcal{F}_{\mathsf{cl},q} = \operatorname{span}_{\mathbb{R}}(\mathsf{d}q^1(q), \dots, \mathsf{d}q^r(q)).$$

- Write $\alpha_{\mathsf{cl}} = \underbrace{\mathbb{G}_{\mathsf{cl},ij}\mathbb{G}_{\mathsf{ol}}^{jk}\frac{\partial V_{\mathsf{ol}}}{\partial q^k}}_{\alpha_{\mathsf{cl},i}}\mathsf{d}q^i.$
- The potential shaping partial differential equation has a solution if

$$\frac{\partial \alpha_{\mathsf{cl},a}}{\partial q^b} = \frac{\partial \alpha_{\mathsf{cl},b}}{\partial q^a}, \qquad a, b \in \{r+1, \dots, n\}.$$

- If $\bar{V}_{\mathsf{cl}}$ is some solution to the potential shaping partial differential equation, then any other solution has the form

$$V_{\mathsf{cl}}(q^1, \dots, q^n) = \bar{V}_{\mathsf{cl}}(q^1, \dots, q^n) + F(q^1, \dots, q^r).$$

4. Discussion

- The proof of the existence part of the theorem is not constructive. It merely tells you that there are no obstructions to constructing a Taylor series solution order-by-order.
- Note that the integrability condition for potential shaping is a condition on $\alpha_{\mathsf{cl}} = \Lambda_{\mathsf{cl}}^{-1} \circ \boldsymbol{d}V_{\mathsf{ol}}$. This is dependent on $\mathbb{G}_{\mathsf{cl}}$.
  The point: A bad design for $\mathbb{G}_{\mathsf{cl}}$ can make it impossible to do *any* potential energy shaping.
- Note that we require integrability of $\mathcal{F}_{\mathsf{cl}} = \Lambda_{\mathsf{cl}}^{-1}(\mathcal{F})$. This is dependent on $\mathbb{G}_{\mathsf{cl}}$.
  The point: A bad design for $\mathbb{G}_{\mathsf{cl}}$ can make it impossible to achieve any flexibility in the character of the possible closed-loop potential functions.

5. Next up...

- What is the character of the solutions to the kinetic energy shaping partial differential equation?