An example with interesting controllability and stabilisation properties

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1. The big picture

- The little contribution here is part of a bigger programme: Understand the geometry of controllability and stabilisation.
- Some interesting questions:
  1. When is a system locally controllable?
  2. When is a system locally stabilisable?
  3. When is a system locally stabilisable using C^0-feedback? (Existing topological results are too strong in their hypotheses.)
  4. What is the relationship between controllability and stabilisability?
- The emphasis is on geometric structure rather than analytical (e.g., Lyapunov) methods.

2. An interesting class of systems

- State is $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Equations:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(t)$$
$$\dot{\boldsymbol{y}}(t) = \boldsymbol{Q}(\boldsymbol{x}(t)),$$

where $\boldsymbol{Q}\colon \mathbb{R}^m \to \mathbb{R}^{n-m}$ is quadratic.

- One can show the following:

Theorem 1 *Suppose that the controls for the system take values in a subset* $U \subset \mathbb{R}^m$ *for which* $\mathbf{0} \in \mathrm{int}(\mathrm{conv}(U))$*. Then the system is small-time locally controllable from* $(\mathbf{0}, \mathbf{0})$ *if and only if* $\mathbf{0} \in \mathrm{int}(\mathrm{conv}(\mathrm{image}(\boldsymbol{Q})))$.

- *Questions:*
  1. Is this class of system locally stabilisable using C^0-feedback?
  2. Is this class of system locally stabilisable if and only if it is locally controllable?
- We do not know the answer to either of these questions.
- In this paper we answer the second question for a certain example in this class by explicitly constructing a stabilising feedback.

3. A transformation using homogeneity

- Given the quadratic nature of the system, we seek a closed-loop system invariant under the action of $\mathbb{R}$ on $\mathbb{R}^n \times \mathbb{R}^{n-m}$ given by

$$\Phi(s, (\boldsymbol{x}, \boldsymbol{y})) \mapsto (\mathrm{e}^s \boldsymbol{x}, \mathrm{e}^{2s} \boldsymbol{y}).$$

- Define

$$\begin{aligned} \rho \colon \mathbb{R}^m \times \mathbb{R}^{n-m} &\to \mathbb{R} \\ (\boldsymbol{x}, \boldsymbol{y}) &\mapsto \sqrt{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|} \end{aligned}$$

so that $\rho^{-1}(1)$ is the homogeneous sphere.

- Each orbit of this group action intersects the homogeneous sphere in one place. Invariant closed-loop system means that the closed-loop dynamics maps orbits to orbits.
  $\longrightarrow$ Closed-loop dynamics drop to the homogeneous sphere.

- The projection from $\mathbb{R}^n \times \mathbb{R}^{n-m}$ to the orbit space (i.e., the homogeneous sphere) is

$$\begin{aligned} \pi \colon \mathbb{R}^m \times \mathbb{R}^{n-m} &\to \rho^{-1}(1) \\ (\boldsymbol{x}, \boldsymbol{y}) &\mapsto (\rho(\boldsymbol{x}, \boldsymbol{y})^{-1} \boldsymbol{x}, \rho(\boldsymbol{x}, \boldsymbol{y})^{-2} \boldsymbol{y}) \end{aligned}$$

- The dynamics on $\rho^{-1}(1)$ is defined by

$$(\boldsymbol{x}, \boldsymbol{y}) \mapsto T_{(\boldsymbol{x}, \boldsymbol{y})} \pi(\boldsymbol{u}, \boldsymbol{Q}(\boldsymbol{x})),$$

for $(\boldsymbol{x}, \boldsymbol{y}) \in \rho^{-1}(1)$.

- Grind...

$$\dot{\boldsymbol{x}} = -\tfrac{1}{2}\boldsymbol{x}\hat{\boldsymbol{y}}^T\boldsymbol{Q}(\boldsymbol{x}) + (\boldsymbol{I}_m - \boldsymbol{x}\boldsymbol{x}^T)\boldsymbol{u},$$
$$\dot{\boldsymbol{y}} = (\boldsymbol{I}_{n-m} - \boldsymbol{y}\hat{\boldsymbol{y}}^T)\boldsymbol{Q}(\boldsymbol{x}) - 2\boldsymbol{y}\boldsymbol{x}^T\boldsymbol{u},$$

- Let us simplify further. Note that $\rho^{-1}(1)$ is not a smooth submanifold, being nondifferentiable on the set

$$S = \{(\boldsymbol{x}, \boldsymbol{y}) \in \rho^{-1}(1) \mid \boldsymbol{y} = \boldsymbol{0}\}.$$

- Let us remove S from $\rho^{-1}(1)$, and note that $\rho^{-1}(1) \setminus S$ is diffeomorphic to $\mathbb{D}^m \times \mathbb{S}^{n-m-1}$ with

$$\mathbb{D}^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid \|\boldsymbol{x}\| = 1\}$$

via the diffeomorphism

$$(\boldsymbol{x}, \boldsymbol{y}) \mapsto (\boldsymbol{x}, \hat{\boldsymbol{y}} = \tfrac{\boldsymbol{y}}{\|\boldsymbol{y}\|}).$$

- Grind...

$$\dot{\boldsymbol{x}} = -\tfrac{1}{2}\boldsymbol{x}\hat{\boldsymbol{y}}^T\boldsymbol{Q}(\boldsymbol{x}) + (\boldsymbol{I}_m - \boldsymbol{x}\boldsymbol{x}^T)\boldsymbol{u},$$
$$\dot{\hat{\boldsymbol{y}}} = (\boldsymbol{I}_{n-m} - \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}^T)\frac{\boldsymbol{Q}(\boldsymbol{x})}{1 - \|\boldsymbol{x}\|^2},$$

for $(\boldsymbol{x}, \hat{\boldsymbol{y}}) \in \mathbb{D}^m \times \mathbb{S}^{n-m-1}$.

- *Punchline:* Assuming homogeneity of the closed-loop system gives the previous system, essentially on the homogeneous sphere. We need to understand the dynamics on this sphere and how it implies stability of the closed-loop system.

4. The closed-loop dynamics on the homogeneous sphere

- The idea is that one designs the closed-loop system so that the dynamics on the homogeneous sphere tends to a region where trajectories in the unreduced space tend asymptotically to zero.
- The key to this is...

Proposition 2 *As in the preceding discussion, let* $\boldsymbol{u}\colon \mathbb{D}^m \times \mathbb{S}^{n-m-1} \to \mathbb{R}^m$ *be a state feedback for the reduced system on the homogeneous sphere, and let* $\overline{\boldsymbol{u}}\colon \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ *be the corresponding state feedback for the unreduced system. Let* $(\boldsymbol{x}_0, \hat{\boldsymbol{y}}_0) \in \mathbb{D}^m \times \mathbb{S}^{n-m-1}$ *be an equilibrium point for the reduced closed-loop system and let* $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ *be the associated point in* $\rho^{-1}(1)$*. Then the following statements hold:*

(i) the corresponding closed-loop trajectory through $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ *in the full state space* $\mathbb{R}^m \times \mathbb{R}^{n-m}$ *is given by* $t \mapsto (\mathrm{e}^{\alpha t}\boldsymbol{x}_0, \mathrm{e}^{2\alpha t}\boldsymbol{y}_0)$*, where* $2\alpha \boldsymbol{y}_0 = \boldsymbol{Q}(\boldsymbol{x}_0)$*;*

(ii) the corresponding closed-loop trajectory through $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ *in the full state space* $\mathbb{R}^m \times \mathbb{R}^{n-m}$ *tends to* $(\mathbf{0}, \mathbf{0})$ *if and only if* $\boldsymbol{y}_0^T \boldsymbol{Q}(\boldsymbol{x}_0) < 0$*.*

- *Punchline:* Need for dynamics on homogeneous sphere to tend to regions where $\boldsymbol{y}_0^T \boldsymbol{Q}(\boldsymbol{x}_0) < 0$.

5. Specialisation to an example

- We consider the three-dimensional system

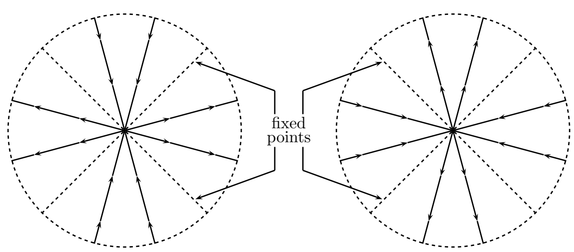
$$\begin{aligned}\dot{x}_1 &= u_1,\\ \dot{x}_2 &= u_2,\\ \dot{y}_1 &= x_1^2 - x_2^2.\end{aligned}$$

- The homogeneous sphere with the singularities removed is diffeomorphic to $\mathbb{D}^2 \times \{-1, 1\}$. The dynamics on it are

$$\begin{aligned}\dot{x}_1 &= \mp \tfrac{1}{2} x_1(x_1^2 - x_2^2) + (1 - x_1^2)u_1 - x_1 x_2 u_2,\\ \dot{x}_2 &= \mp \tfrac{1}{2} x_2(x_1^2 - x_2^2) - x_1 x_2 u_1 + (1 - x_2^2)u_2,\end{aligned}$$

where "$+$" occurs on $\mathbb{D}^2 \times \{-1\}$ and "$-$" occurs on $\mathbb{D}^2 \times \{1\}$.

- The key is the dynamics of the drift vector field:



- According to Proposition 2 we want to steer the system to the regions where the drift vector field points away from the origin. Call these ***good regions***.
- One can show that the control vector fields are linearly independent everywhere except at the boundary of the disk where they degenerate to be tangent to the boundary.

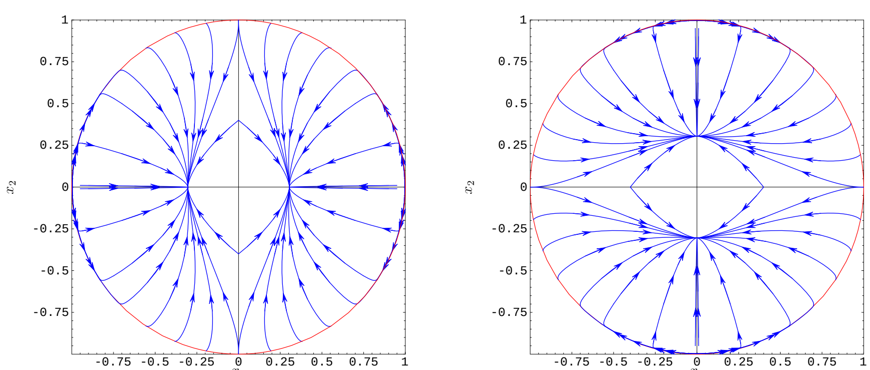
- This leads to the following control strategy.

Procedure 3 Design controls so that on each disk $\mathbb{D}^2 \times \{\pm 1\}$ the closed-loop dynamics have the following properties:

(i) trajectories either leave the disk or tend to an equilibrium point in the good region;

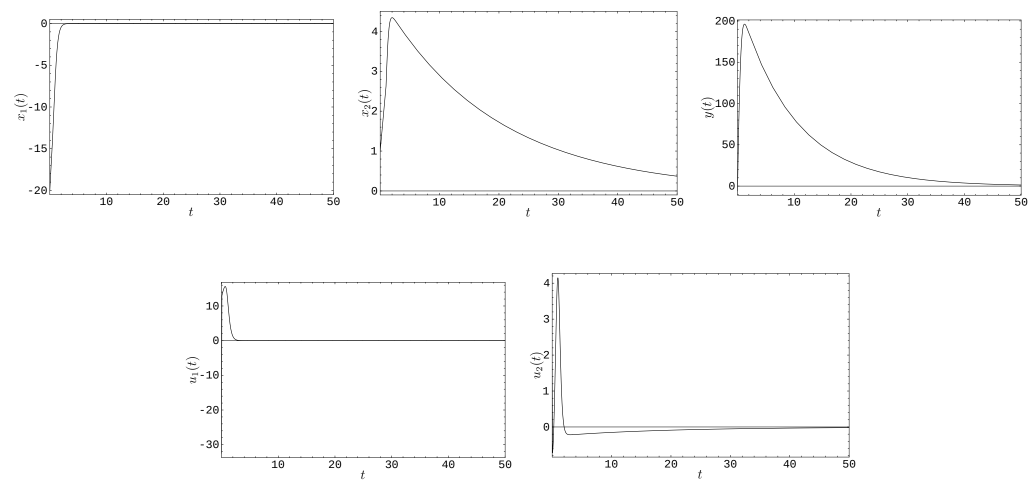
(ii) if a trajectory enters the disk it remains in the disk thereafter.

- It is possible to do this. Here are the closed-loop phase portraits.



- The proof of stability is just a matter of seeing that these phase portraits have the properties in the procedure above.

- Here are the obligatory plots of solutions tending to zero in state space.



- Performance is not that good:
  1. can be improved by modifying the reduced dynamics;
  2. inevitable, to some extent, since this is a hard system to stabilise.

6. What's the point?

- The example system, while fairly simple, cannot be stabilised by general schemes from the literature that we are aware of.
- *By understanding the geometry of the system*, one can nonetheless design a discontinuous stabilising feedback.
- We can also do this for

$$\dot{x}_1 = u_1, \qquad \dot{y}_1 = x_1 x_2,$$
$$\dot{x}_2 = u_2, \qquad \dot{y}_1 = x_1^2 - x_2^2,$$

but the system

$$\dot{x}_1 = u_1, \qquad \dot{y}_1 = x_1 x_2,$$
$$\dot{x}_2 = u_2, \qquad \dot{y}_2 = x_1 x_3,$$
$$\dot{x}_3 = u_3, \qquad \dot{y}_3 = x_2 x_3$$

seems fundamentally more difficult.