Some results on energy shaping feedback

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13/08/2007

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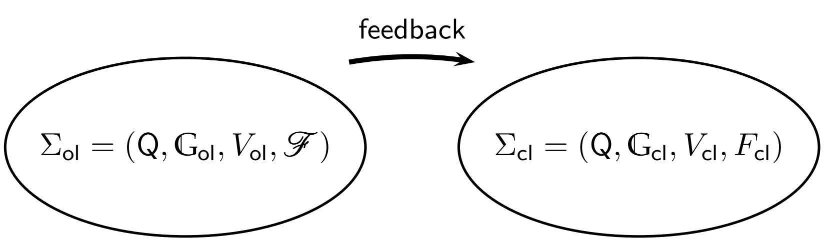
The problem

- Given an open-loop mechanical control system $\Sigma_{\mathsf{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}})$:



- The closed-loop system $\Sigma_{\mathsf{cl}} = (\mathsf{Q}, \mathbb{G}_{\mathsf{cl}}, V_{\mathsf{cl}})$ should have some desired properties, e.g., an equilibrium point q_0 should be stable.

- In equations:
  Given:
  $$\overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}_{\mathsf{ol}}^{\sharp} \circ \boldsymbol{d}V_{\mathsf{ol}}(\gamma(t)) + \sum_{a=1}^{m} u^a(t)\mathbb{G}_{\mathsf{ol}}^{\sharp} \circ F^a(\gamma(t)).$$
  Find: Feedback controls $u_{\mathsf{shp}}\colon \mathsf{TQ} \to \mathbb{R}^m$ such that the closed-loop system has governing equations
  $$\overset{\mathbb{G}_{\mathsf{cl}}}{\nabla}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}_{\mathsf{cl}}^{\sharp} \circ \boldsymbol{d}V_{\mathsf{cl}}(\gamma(t)) + \mathbb{G}_{\mathsf{cl}}^{\sharp} \circ F_{\mathsf{cl}}(\gamma'(t)).$$
- *Form of F_{cl}:* $F_{\mathsf{cl}} = F_{\mathsf{cl,diss}} + F_{\mathsf{cl,gyr}}$ where
  1. $F_{\mathsf{cl,diss}}$ is a dissipative force and
  2. $F_{\mathsf{cl,gyr}}$ is a quadratic gyroscopic force.

- *Recall:* A **quadratic gyroscopic force** is of the form
  $$\langle F_{\mathsf{gyr}}(v_q); w_q\rangle = B_{\mathsf{gyr}}(w_q, v_q, v_q),$$
  where B_{gyr} is a $(0,3)$-tensor field satisfying
  $$B_{\mathsf{gyr}}(u_q, v_q, w_q) = -B_{\mathsf{gyr}}(v_q, u_q, w_q)$$
  (denote $F_{\mathsf{gyr}}(v_q) = B_{\mathsf{gyr}}^{\flat}(v_q)$).
  Punchline: Preserves energy.

- *Assumed procedure:*
  1. Find closed-loop (kinetic energy)/(quadratic gyroscopic force):
  $$\mathbb{G}_{\mathsf{ol}}^{\sharp} \circ F_{\mathsf{kin}}(\gamma'(t)) = \overset{\mathbb{G}_{\mathsf{cl}}}{\nabla}_{\gamma'(t)}\gamma'(t) + \mathbb{G}_{\mathsf{cl}}^{\sharp} \circ B_{\mathsf{cl,gyr}}^{\flat}(\gamma'(t)) - \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t)$$
  2. Find closed-loop potential energy:
  $$F_{\mathsf{pot}}(\gamma(t)) = \underbrace{\mathbb{G}_{\mathsf{ol}}^{\flat} \circ \mathbb{G}_{\mathsf{cl}}^{\sharp}}_{\Lambda_{\mathsf{cl}}} \circ \boldsymbol{d}V_{\mathsf{cl}}(\gamma(t)) - \boldsymbol{d}V_{\mathsf{ol}}(\gamma(t)).$$
  3. Find closed-loop control:
  $$\sum_{a=1}^{m} u_{\mathsf{shp}}^{a}(v_q)\mathbb{G}_{\mathsf{ol}}^{\sharp} \circ F^{a}(q) = -F_{\mathsf{kin}}(v_q) - F_{\mathsf{pot}}(q).$$
- *Today:* Ignore dissipative forces.

Objectives of approach

- What are the possible closed loop energies,
$$E_{\mathsf{cl}}(v_q) = \tfrac{1}{2}\mathbb{G}_{\mathsf{cl}}(v_q) + V_{\mathsf{cl}}(q)?$$
- For stabilisation: Want $\operatorname{Hess} V_{\mathsf{cl}}(q_0) > 0$?
- Main limitation for stabilisation: only works for systems that are linearly stabilizable, i.e., doesn't work for "hard" systems (i.e., requiring discontinuous feedback) $\Longrightarrow$ if there are benefits, they are global in nature.

Partial literature review and comments

- Potential shaping:

1. M. Takegaki and S. Arimoto, "A new feedback method for dynamic control of manipulators," *Trans. ASME Ser. G J. Dynamic Systems Measurement Control*, vol. 103, no. 2, pp. 119–125, 1981.
2. A. J. van der Schaft, "Stabilization of Hamiltonian systems," *Nonlinear Anal. TMA*, vol. 10, no. 10, pp. 1021–1035, 1986.
3. A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and G. Sánchez de Alvarez, "Stabilization of rigid body dynamics by internal and external torques," *Automatica—J. IFAC*, vol. 28, no. 4, pp. 745–756, 1992.

- Hamiltonian approach (IDA-PBC):

4. R. Ortega, M. W. Spong, F. Gómez-Estern, and G. Blankenstein, "Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment," *IEEE Trans. Automat. Control*, vol. 47, no. 8, pp. 1218–1233, 2002.

- Lagrangian approach with symmetry:

5. A. M. Bloch, N. E. Leonard, and J. E. Marsden, "Controlled Lagrangians and the stabilization of mechanical systems. I. The first matching theorem," *IEEE Trans. Automat. Control*, vol. 45, no. 12, pp. 2253–2270, 2000.
6. A. M. Bloch, D. E. Chang, N. E. Leonard, and J. E. Marsden, "Controlled Lagrangians and the stabilization of mechanical systems. II. Potential shaping," *IEEE Trans. Automat. Control*, vol. 46, no. 10, pp. 1556–1571, 2001.

- Equivalence of Lagrangian and Hamiltonian setting:
  7. D. E. Chang, A. M. Bloch, N. E. Leonard, J. E. Marsden, and C. A. Woolsey, "The equivalence of controlled Lagrangian and controlled Hamiltonian systems," *ESAIM Contrôle Optim. Calc. Var.*, vol. 8, pp. 393–422, 2002.
  8. G. Blankenstein, R. Ortega, and A. J. van der Schaft, "The matching conditions of controlled Lagrangians and IDA-passivity based control," *Internat. J. Control*, vol. 75, no. 9, pp. 645–665, 2002.
- Geometric formulation and weak integrability results:
  9. D. R. Auckly and L. V. Kapitanski, "On the λ-equations for matching control laws," *SIAM J. Control Optim.*, vol. 41, no. 5, pp. 1372–1388, 2002.
  10. D. R. Auckly, L. V. Kapitanski, and W. White, "Control of nonlinear underactuated systems," *Comm. Pure Appl. Math.*, vol. 53, no. 3, pp. 354–369, 2000.

- Extension to general Lagrangians:
  11. J. Hamberg, "General matching conditions in the theory of controlled Lagrangians," in *Proceedings of the 38th IEEE CDC*. Phoenix, AZ: IEEE, Dec. 1999, pp. 2519–2523.
  12. J. Hamberg, "Simplified conditions for matching and for generalized matching in the theory of controlled Lagrangians," in *Proceedings of the 2000 American Control Conference*, Chicago, IL, June 2000, pp. 3918–3923.
- Extension to nonholonomic systems:
  13. D. V. Zenkov, "Matching and stabilization of the unicycle with rider," in *Proc. IFAC Workshop on Lagrangian and Hamiltonian Methods in Nonlinear Control*, Princeton, NJ, March 2000, pp. 187–188.
- Linear systems:
  14. D. V. Zenkov, "Matching and stabilization of linear mechanical systems," in *Proceedings of MTNS 2002*, South Bend, IN, Aug. 2002.

A theorem on potential energy shaping

The setup

- We have an open-loop system $(\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \{F^1, \ldots, F^m\})$ and have applied a control to shape the kinetic energy to $\mathbb{G}_{\mathsf{cl}}$.
- Define $\Lambda_{\mathsf{cl}} = \mathbb{G}_{\mathsf{ol}}^\flat \circ \mathbb{G}_{\mathsf{cl}}^\sharp$.
- Let $\mathcal{F}$ be the codistribution generated by $\{F^1, \ldots, F^m\}$ (control forces are thus $\mathcal{F}$-valued). Assume constant rank.
- Let $\mathcal{F}_{\mathsf{cl}}$ be the codistribution $\Lambda_{\mathsf{cl}}^{-1}(\mathcal{F})$.

Lemma 1 *Given:* $\mathbb{G}_{\mathsf{ol}}$, V_{ol}, $\mathbb{G}_{\mathsf{cl}}$, *and* $\mathcal{F}$.
A force F *taking values in* $\mathcal{F}$ *gives a closed-loop potential* V_{cl} *if and only if*

$$F(q) = \Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V_{\mathsf{cl}}(q) - \boldsymbol{d}V_{\mathsf{ol}}(q), \qquad q \in \mathsf{Q}.$$

The potential energy shaping partial differential equation

- Define $\mathsf{Q}_{\mathbb{R}} = \mathsf{Q} \times \mathbb{R}$ and $\pi \colon \mathsf{Q}_{\mathbb{R}} \to \mathsf{Q}$ by $\pi(q, V) = q$.
- A potential function V defines a section of $\mathsf{Q}_{\mathbb{R}}$: $q \mapsto (q, V(q))$.
- We have the map $\Phi_{\boldsymbol{d}} \colon \mathsf{J}^1\mathsf{Q}_{\mathbb{R}} \to \mathsf{T}^*\mathsf{Q}$ satisfying $\Phi_{\boldsymbol{d}}(j_1V(q)) = \boldsymbol{d}V(q)$.
- Abbreviate $\alpha_{\mathsf{cl}} = \Lambda_{\mathsf{cl}}^{-1} \circ \boldsymbol{d}V_{\mathsf{ol}}$.
- Let $\pi_{\mathcal{F}_{\mathsf{cl}}} \colon \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}/\mathcal{F}_{\mathsf{cl}}$ be the canonical projection.
- Define

$$\mathsf{R}_{\mathsf{pot}} = \{j_1V(q) \in \mathsf{J}^1\mathsf{Q}_{\mathbb{R}} \mid \pi_{\mathcal{F}_{\mathsf{cl}}} \circ \Phi_{\boldsymbol{d}}(j_1V(q)) = \pi_{\mathcal{F}_{\mathsf{cl}}} \circ \alpha_{\mathsf{cl}}(q)\}.$$

Proposition 1 *A section F of $\mathcal{F}$ is a potential energy shaping feedback if and only if $F = \Lambda_{\mathsf{cl}} \circ \boldsymbol{d}V - \boldsymbol{d}V_{\mathsf{ol}}$ for a solution V to $\mathsf{R}_{\mathsf{pot}}$.*

- What's the point of all the fanciness?
- You get a partial differential equation in the framework for applying the Goldschmidt[1] theory for integrability of partial differential equations.

[1] *J. Differential Geom.*, **1**, 269–307, 1967

The statement

- Let $\mathsf{I}_2(\mathcal{F}_{\mathsf{cl}})$ be the two-forms in the algebraic ideal generated by $\mathcal{F}_{\mathsf{cl}}$.

Theorem 1 *Let $(\mathsf{Q}, \mathbb{G}_{\mathsf{ol}}, V_{\mathsf{ol}}, \mathscr{F})$ be an analytic simple mechanical control system and let $\mathbb{G}_{\mathsf{cl}}$ be an analytic Riemannian metric. Let $j_1V(q_0) \in \mathsf{R}_{\mathsf{pot}}$. Assume that q_0 is a regular point for $\mathcal{F}$ and that $\mathcal{F}_{\mathsf{cl}}$ is integrable in a neighbourhood of q_0. Then the following statements are equivalent:*

(i) there exists a neighbourhood $\mathcal{U}$ of q_0 and an analytic potential energy shaping feedback F defined on $\mathcal{U}$ which satisfies $\Phi_{\boldsymbol{d}}(j_1V(q_0)) = F_{\mathsf{cl}}(q_0) + \alpha_{\mathsf{cl}}(q_0)$;

(ii) there exists a neighbourhood $\mathcal{U}$ of q_0 such that $\boldsymbol{d}\alpha_{\mathsf{cl}}(q) \in \mathsf{I}(\mathcal{F}_{\mathsf{cl},q})$ for each $q \in \mathcal{U}$.

Moreover, if $V_{\mathsf{cl},1}$ and $V_{\mathsf{cl},2}$ are two closed-loop potential functions, then $\boldsymbol{d}(V_{\mathsf{cl},1} - V_{\mathsf{cl},2})(q) \in \mathcal{F}_{\mathsf{cl},q}$ for each $q \in \mathsf{Q}$.

The working version

- If $\mathcal{F}_{\mathrm{cl}}$ is not integrable, replace it with the largest integrable codistribution contained in it.
- Since $\mathcal{F}_{\mathrm{cl}}$ is integrable choose coordinates $(q^1, \ldots, q^n)$ for Q such that

$$\mathcal{F}_{\mathrm{cl},q} = \mathrm{span}_{\mathbb{R}}(\mathrm{d}q^1(q), \ldots, \mathrm{d}q^r(q)).$$

- Write $\alpha_{\mathrm{cl}} = \underbrace{\mathbb{G}_{\mathrm{cl},ij}\mathbb{G}_{\mathrm{ol}}^{jk}\frac{\partial V_{\mathrm{ol}}}{\partial q^k}}_{\alpha_{\mathrm{cl},i}}\mathrm{d}q^i$.
- The potential shaping partial differential equation has a solution if

$$\frac{\partial \alpha_{\mathrm{cl},a}}{\partial q^b} = \frac{\partial \alpha_{\mathrm{cl},b}}{\partial q^a}, \qquad a, b \in \{r+1, \ldots, n\}.$$

- If $\bar{V}_{\mathrm{cl}}$ is some solution to the potential shaping partial differential equation, then any other solution has the form

$$V_{\mathrm{cl}}(q^1, \ldots, q^n) = \bar{V}_{\mathrm{cl}}(q^1, \ldots, q^n) + F(q^1, \ldots, q^r).$$

Discussion

- The proof of the existence part of the theorem is not constructive. It merely tells you that there are no obstructions to constructing a Taylor series solution order-by-order.
- Note that the integrability condition for potential shaping is a condition on $\alpha_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1} \circ \boldsymbol{d}V_{\mathrm{ol}}$. This is dependent on $\mathbb{G}_{\mathrm{cl}}$.
  The point: A bad design for $\mathbb{G}_{\mathrm{cl}}$ can make it impossible to do *any* potential energy shaping.
- Note that we require integrability of $\mathcal{F}_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1}(\mathcal{F})$. This is dependent on $\mathbb{G}_{\mathrm{cl}}$.
  The point: A bad design for $\mathbb{G}_{\mathrm{cl}}$ can make it impossible to achieve any flexibility in the character of the possible closed-loop potential functions.

An affine connection formulation of kinetic energy shaping

- For a Riemannian metric $\mathbb{G}$, define $\mathrm{KE}_{\mathbb{G}} \colon \mathsf{TQ} \to \mathbb{R}$ by $\mathrm{KE}_{\mathbb{G}}(v_q) = \frac{1}{2}\mathbb{G}(v_q, v_q)$.
- An affine connection ∇ is **$\mathbb{G}$-energy-preserving** if $\mathscr{L}_{\gamma''(t)}\mathrm{KE}_{\mathbb{G}}(\gamma'(t)) = 0$ for every geodesic γ of ∇.

Lemma 2 *∇ is $\mathbb{G}$-energy preserving if and only if $\mathrm{Sym}(\nabla\mathbb{G}) = 0$.*

Theorem 2 *Given: $\mathbb{G}_{\mathsf{ol}}$ and $\mathcal{F}$.*
The solutions to the following problems are in 1–1 correspondence:

(i) when does there exist $\mathbb{G}_{\mathsf{cl}}$ and a gyroscopic tensor B_{cl} such that

$$\overset{\mathbb{G}_{\mathsf{cl}}}{\nabla}_{\gamma'(t)}\gamma'(t) + \mathbb{G}_{\mathsf{cl}}^{\sharp} \circ B_{\mathsf{cl}}^{\flat}(\gamma'(t)) - \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t) \in \mathbb{G}_{\mathsf{ol}}^{\sharp}(\mathcal{F});$$

(ii) when does there exist $\mathbb{G}_{\mathsf{cl}}$ and a $\mathbb{G}_{\mathsf{cl}}$-energy preserving connection $\overset{\mathsf{cl}}{\nabla}$ such that

$$\overset{\mathsf{cl}}{\nabla}_{\gamma'(t)}\gamma'(t) - \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}_{\gamma'(t)}\gamma'(t) \in \mathbb{G}_{\mathsf{ol}}^{\sharp}(\mathcal{F}).$$

The kinetic energy shaping partial differential equation

Geometric formulation of partial differential equation

- Recall that the set of torsion-free affine connections on Q is an affine subbundle

$$\begin{aligned}\mathrm{Aff}_0(\mathsf{Q}) = \{\Gamma \in \mathsf{T}^*\mathsf{Q} \otimes \mathsf{J}^1\mathsf{TQ} \mid \ & \Gamma \circ \pi_0^1 = \mathrm{id}_{\mathsf{TQ}}, \\ & (j_1 Y - \Gamma(Y))(X) - (j_1 X - \Gamma(X))(Y) = [X, Y]\}\end{aligned}$$

  modelled on $\mathrm{S}^2(\mathsf{T}^*\mathsf{Q}) \otimes \mathsf{TQ}$.

- Let $\mathsf{Y}_{\mathsf{KE}} = \{(\Gamma, \mathbb{G}) \in \mathrm{Aff}_0(\mathsf{Q}) \times \mathrm{S}^2_+(\mathsf{T}^*\mathsf{Q}) \mid \Gamma \text{ is } \mathbb{G}\text{-energy preserving}\}$.
- Define $\Phi_{\mathsf{LC}} \colon \mathsf{J}^1\mathrm{S}^2_+(\mathsf{T}^*\mathsf{Q}) \to \mathrm{Aff}_0(\mathsf{Q})$ by $\Phi_{\mathsf{LC}}(j_1\mathbb{G}) = \overset{\mathbb{G}}{\nabla}$.
- Define the quasilinear partial differential equation

$$\mathsf{R}_{\mathsf{kin}} = \{(j_1\Gamma(q), j_1\mathbb{G}(q)) \in \mathsf{J}^1\mathsf{Y} \mid \pi_{\mathcal{F}}(\Gamma(q) - \Phi_{\mathsf{KE}}(j_1\mathbb{G}(q))) = 0\},$$

  where $\pi_{\mathcal{F}} \colon \mathrm{S}(\mathsf{T}^*\mathsf{Q}) \otimes \mathsf{TQ} \to \mathrm{S}(\mathsf{T}^*\mathsf{Q}) \otimes \mathsf{TQ}/\mathbb{G}_{\mathsf{ol}}^{\sharp}(\mathcal{F})$ is the canonical projection.

An observation

- Define two subsets of $\mathrm{Aff}_0(\mathsf{Q})$:

$$\begin{aligned}&\mathrm{Aff}_0(\mathsf{Q}, \mathcal{F}, \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}) = \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla} + \mathrm{S}^2(\mathsf{T}^*\mathsf{M}) \otimes \mathrm{coann}(\mathcal{F}), \\ &\mathrm{EP}(\mathsf{Q}) = \{\nabla \in \mathrm{Aff}_0(\mathsf{Q}) \mid \nabla \text{ is } \mathbb{G}\text{-energy preserving for some } \mathbb{G}\}.\end{aligned}$$

- The solutions $(\overset{\mathsf{cl}}{\nabla}, \mathbb{G}_{\mathsf{cl}})$ to $\mathsf{R}_{\mathsf{kin}}$ are then described by asking that

$$\overset{\mathsf{cl}}{\nabla} \in \mathrm{Aff}_0(\mathsf{Q}, \mathcal{F}, \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla}) \cap \mathrm{EP}(\mathsf{Q}).$$

- $\mathrm{Aff}_0(\mathsf{Q}, \mathcal{F}, \overset{\mathbb{G}_{\mathsf{ol}}}{\nabla})$ is easy to understand.
- What about $\mathrm{EP}(\mathsf{Q})$?
- And when $\nabla \in \mathrm{EP}(\mathsf{Q})$ what does $\{\mathbb{G} \mid \mathrm{Sym}(\nabla\mathbb{G}) = 0\}$ look like?

Relationship to an inverse problem in calculus of variations

- Consider the following subset of $\mathrm{EP}(\mathsf{Q})$:

$$\mathrm{LC}(\mathsf{Q}) = \{\nabla \in \mathrm{Aff}_0(\mathsf{Q}) \mid \nabla \text{ is the Levi-Civita connection for some } \mathbb{G}\}.$$

- The problem was initially investigated by Eisenhart and Veblen[1] who give necessary conditions and a sufficient condition with strong hypotheses.
- Comparison of problems:

$\mathrm{LC}(\mathsf{Q})$	$\mathrm{EP}(\mathsf{Q})$
$\nabla\mathbb{G} = 0$ has solution	$\mathrm{Sym}(\nabla\mathbb{G}) = 0$ has solution

- The Eisenhart and Veblen problem is "nice:" it has an involutive symbol.
- The symbol for our generalisation is not involutive ⟶ work to do here.

[1] *Proceedings of the National Academy of Sciences of the United States of America*, **8**, 19–23, 1922

Summary

- The method of energy shaping has been applied in certain cases, sometimes with some generality. However. . .
- The question, "If I give you a system, can you determine whether it can be stabilised using energy shaping" remains unresolved.