# Differential geometric methods for the control of mechanical systems

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# The objective of the talk

- To illustrate where some fairly sophisticated mathematics has been used to solve (hopefully somewhat interesting) problems that may be difficult, or impossible, to solve otherwise.
- *Collaborators:* Francesco Bullo, Bahman Gharesifard, Kevin Lynch, Richard Murray, David Tyner.
- *Relies on work by:* Suguru Arimoto, Guido Blankenstein, Anthony Bloch, Dong Eui Chang, Hubert Goldschmidt, Fabio Gómez-Estern, Velimir Jurdjevic, Naomi Leonard, Jerrold Marsden, Romeo Ortega, Mark Spong, Héctor Sussmann, Morikazu Takegaki, Arjan van der Schaft.

# Some toy problems to keep in mind
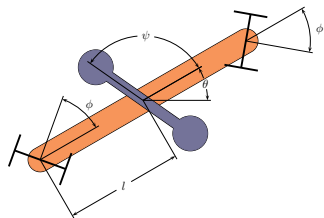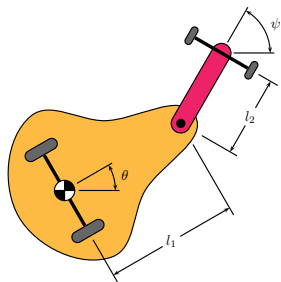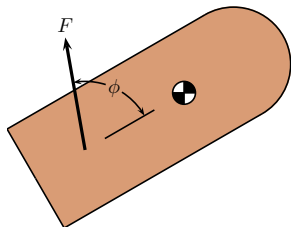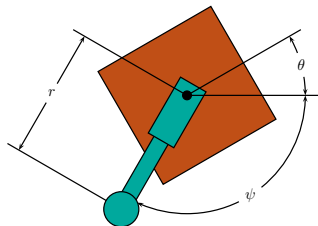
# Mechanical systems: mathematical modelling

- *Question:* What is the mathematical structure of the equations governing the motion of a mechanical system?
- We will use the Euler–Lagrange equations.
- We begin with the kinetic energy Lagrangian.
- Expressed in ("generalised") coordinates $(q^1, \ldots, q^n)$ this Lagrangian is

$$L = \sum_{i,j=1}^{n} \frac{1}{2} \mathbb{G}_{ij}(q)\dot{q}^i\dot{q}^j.$$

- Here $\mathbb{G}_{ij}(q)$, $i,j = 1, \ldots, n$, are the components of a symmetric $n \times n$ matrix which represents the inertial properties of the system.
- $\mathbb{G}$ is the ***kinetic energy metric***.

# Mechanical systems: mathematical modelling

- For a system with kinetic energy determined by the kinetic energy metric $\mathbb{G}$ and acted upon by no external forces, the following statements are equivalent for a curve $\gamma$ in configuration space:

  1. $\gamma$ satisfies the Euler–Lagrange equations,

  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad i \in \{1, \ldots, n\},$$

  where $L$ is the kinetic energy Lagrangian;

  2. $\gamma$ satisfies

  $$\ddot{q}^i + \sum_{j,k=1}^{n} \overset{\mathbb{G}}{\Gamma}{}^i_{jk} \dot{q}^j \dot{q}^k = 0, \qquad i \in \{1, \ldots, n\},$$

  where

  $$\overset{\mathbb{G}}{\Gamma}{}^i_{jk} = \frac{1}{2} \mathbb{G}^{il}\left(\frac{\partial \mathbb{G}_{lj}}{\partial q^k} + \frac{\partial \mathbb{G}_{lk}}{\partial q^j} - \frac{\partial \mathbb{G}_{jk}}{\partial q^l}\right), \qquad i, j, k \in \{1, \ldots, n\};$$

  3. $\gamma$ is a geodesic for the Levi-Civita affine connection: $\overset{\mathbb{G}}{\nabla}_{\gamma'(t)} \gamma'(t) = 0$.

# Mechanical systems: mathematical modelling

- Now let's add forces. A force is modelled by a bundle map $F\colon \mathbb{R} \times \mathsf{T}Q \to \mathsf{T}^*Q$.
- For a system with kinetic energy determined by the kinetic energy metric $\mathbb{G}$ and acted upon by no external forces, the following statements are equivalent for a curve $\gamma$ in configuration space:

  1. $\gamma$ satisfies the Euler–Lagrange equations,

  $$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}^i}\Big) - \frac{\partial L}{\partial q^i} = F_i; \qquad i \in \{1,\ldots,n\};$$

  2. $\gamma$ satisfies

  $$\ddot{q}^i + \sum_{j,k=1}^{n} \overset{\mathbb{G}}{\Gamma}{}^i_{jk}\dot{q}^j\dot{q}^k = 0, \qquad i \in \{1,\ldots,n\};$$

  3. $\gamma$ satisfies the forced geodesic equations:
  $$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = \mathbb{G}^{-1} \circ F(t,\gamma'(t)).$$

# What have we done?

- We have a compact (and well-known) representation of the equations governing the motion of a mechanical system, and a prominent rôle is played by the Levi-Civita connection associated with the kinetic energy metric.
- *Fact:* By relaxing the assumption that the affine connection be the Levi-Civita connection associated with the kinetic energy metric, we may include systems with nonholonomic (e.g., rolling) constraints. (This is not obvious.)
- *Question:* Can we do anything interesting with the structure in our representation of the equations of motion?
- *Answer:* I think so, in control theory.

# Control theory for mechanical systems

- In control theory we have control over some of the external forces.
- Thus we write the external force $F$ as

$$F = F_{\text{ext}} + \sum_{a=1}^{m} u^a F^a,$$

  where $F_{\text{ext}}$ represents uncontrolled forces and the total control force is $\sum_{a=1}^{m} u^a F^a$, i.e., the control force is a linear combination of forces $F^1, \ldots, F^m$.

- *Assumption:* $F^1, \ldots, F^m$ depend only on configuration, and not on time or velocity.

# Control theory for mechanical systems

- The governing equations we consider are then

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = Y_{\text{ext}}(t, \gamma'(t)) + \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t)),$$
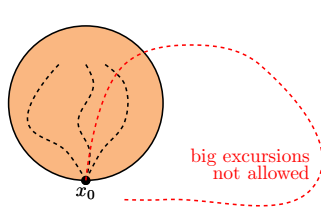
  where $Y_{\text{ext}} = \mathbb{G}^{-1} \circ F_{\text{ext}}$ and $Y_a = \mathbb{G}^{-1} \circ F^a$, $a = 1, \ldots, m$.
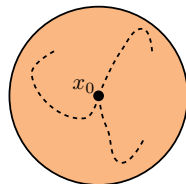
- *Questions:*
  1. *Controllability:* Can a state $x_2$ be reached from a state $x_1$ by a suitable control $u$?
  2. *Stabilisability:* Can a state $x_0$ be made a stable equilibrium point for the system after a suitable control $u$ has been prescribed?
  3. *Motion planning: Design* a control steering $x_1$ to $x_2$.
  4. *Stabilisation: Design* a control $u$ that renders $x_0$ a stable equilibrium point.

# Controllability of mechanical systems: Definitions

$$\overset{\mathbb{G}}{\nabla}_{\gamma'(t)}\gamma'(t) = Y_{\mathsf{ext}}(t, \gamma'(t)) + \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t))$$



*accessibility*

big excursions
not allowed

$x_0$

*controllability*

- Accessibility (does the set of points reachable from $x_0$ have a nonempty interior?) is easily decidable.
- Controllability (is $x_0$ in the interior of its own reachable set?) is very difficult to decide.

# Controllability of mechanical systems: Illustration

- Controllability is only an interesting problem for underactuated systems; this excludes the "typical" robot. An example illustrates how controllability works.
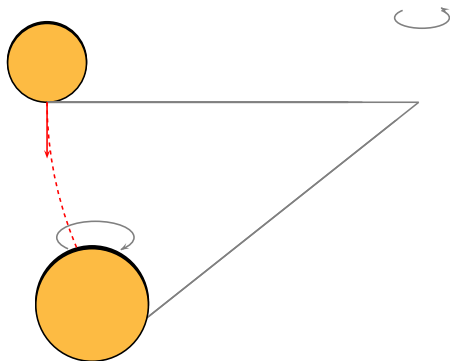
# Controllability of mechanical systems: Results

- We consider systems with no external forces:

$$\nabla_{\gamma'(t)} \gamma'(t) = \sum_{a=1}^{m} u^a(t) Y(\gamma(t))$$

  (note we do not require $\nabla$ to be the Levi-Civita connection).

- Call this an **affine connection control system**.
- Let $Y \subset TQ$ be the distribution generated by the vector fields $Y_1, \ldots, Y_m$:

$$Y_q = \mathrm{span}_{\mathbb{R}}(Y_1(q), \ldots, Y_m(q)).$$

- Define the **symmetric product** $\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$.
- Let $\mathrm{Sym}^{(\infty)}(Y)$ be the smallest distribution containing $Y$ and closed under symmetric product.
- Let $\mathrm{Lie}^{(\infty)}(\mathrm{Sym}^{(\infty)}(Y))$ be the smallest distribution containing $\mathrm{Sym}^{(\infty)}(Y)$ and closed under Lie bracket.

# Controllability of mechanical systems: Results

## Theorem (Accessibility for affine connection control systems)

*For an analytic affine connection control system, the following statements are equivalent:*

(i) *the system is accessible from* $q_0$;

(ii) $\mathrm{Lie}^{(\infty)}(\mathrm{Sym}^{(\infty)}(Y))_{q_0} = \mathsf{T}_{q_0}\mathsf{Q}$.

- *Geometric aside:* What is the meaning of the symmetric product?
- Say a distribution $\mathsf{D} \subset \mathsf{TQ}$ is **geodesically invariant** if, for a geodesic $\gamma$ such that $\gamma'(t_0) \in \mathsf{D}_{\gamma(t_0)}$, $\gamma(t) \in \mathsf{D}_{\gamma(t)}$ for every $t$.

## Theorem (Meaning of symmetric product)

*For a distribution* $\mathsf{D}$ *the following statements are equivalent:*

(i) $\mathsf{D}$ *is geodesically invariant;*

(ii) $\langle X : Y \rangle$ *is* $\mathsf{D}$*-valued for* $\mathsf{D}$*-valued vector fields* $X$ *and* $Y$.

# Controllability of mechanical systems: Results

- Controllability (as opposed to accessibility) results exist, but are a little complicated (and frankly unsatisfying) to state.
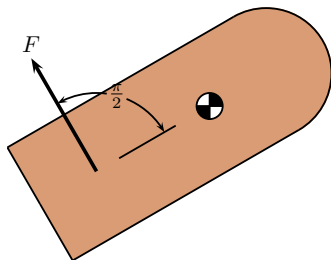- So let's just look at some examples.

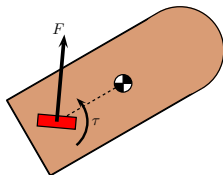# Controllability of mechanical systems: Examples
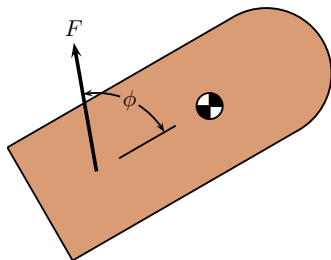


- Hovercraft system:
  1. *Question:* Is the system accessible?
  2. *Answer:* Yes (easy).
  3. *Question:* Is the system controllable?
  4. *Answer:* Yes (a little harder).

# Controllability of mechanical systems: Examples



- Now suppose that the fan cannot rotate.
    1. *Question:* Is the system accessible?
    2. *Answer:* Yes (easy).
    3. *Question:* Is the system controllable?
    4. *Answer:* No, at least not locally (nontrivial).

# Controllability of mechanical systems: Examples



- Change the model by adding inertia to the fan.
  1. *Question:* Is the system accessible?
  2. *Answer:* Yes (easy).
  3. *Question:* Is the system controllable?
  4. *Answer:* No, at least not locally (getting really difficult now).

# The punchline

- By slight alterations of the problem, a somewhat simple problem can be made very hard. To determine the answers to some of the controllability questions, difficult general theorems had to be proved.
- *So what?* Can the affine connection actually be used to *solve* a problem?
- Let's look at the motion planning problem.

# Motion planning: Motivation
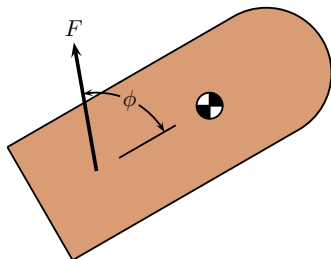


- Imagine trying to steer the hovercraft from one configuration at rest to another.
- We know this is possible (we answered the controllability question in the affirmative). But *how* can we do this?

# Motion planning: Results

- For the affine connection control system

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t))$$

  one can pose a natural question: What are those vector fields whose integral curves we can follow with an arbitrary parameterisation?

- Call these **decoupling vector fields**.

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### Theorem (Characterisation of decoupling vector fields)

*For a vector field $X$ the following statements are equivalent:*

*(i) $X$ is a decoupling vector field;*

*(ii) $X(q) \in \mathsf{Y}_q$ and $\nabla_X X(q) \in \mathsf{Y}_q$ for every $q \in \mathsf{Q}$.*

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# Motion planning: Results

- Compare velocity (kinematic) and acceleration (dynamic) control:

$$\gamma'(t) = \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t)) \qquad \nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t)).$$

- Somewhat imprecisely, these systems are **equivalent** if they have the same trajectories $\gamma$.

### Theorem (Equivalence of kinematic and dynamic systems)

*For an affine connection control system, the following statements are equivalent:*

*(i) the kinematic and dynamic systems are equivalent;*

*(ii) $\langle X : Y \rangle$ is $Y$-valued for $Y$-valued vector fields $X$ and $Y$.*

# Motion planning: Illustrations

- The preceding theorems sometimes allow one to construct explicit solutions to the motion planning problem.
- For the planar body:
  Planar body motion 1
  Planar body motion 2
  Planar body motion plan

  Another flavour of motion planner
  Yet another flavour of motion planner

- For the snakeboard:
  Snakeboard motion plan 1
  Snakeboard motion plan 2

# Stabilisation using energy shaping: Motivation

- We are now thinking about mechanical systems for which the external force is solely provided by means of a potential function.
- We are interested here in the stabilisation problem. For systems with potential forces, equilibria are points where the derivative of the potential function is zero. An equilibrium is stable if it is a minimum of the potential function and unstable if it is a maximum of the potential function:
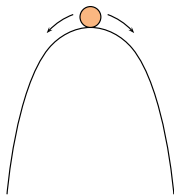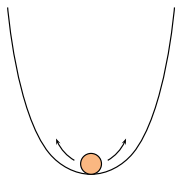
# Stabilisation using energy shaping: Motivation

- *Problem:* Using control, can we take a system with an unstable equilibrium and make it stable by altering the potential function to have a minimum at the desired point?

- For example, one can imagine the classical problem of stabilising the cart/pendulum system with the pendulum up:
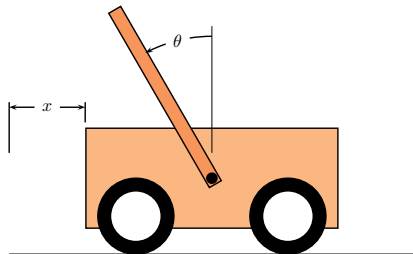


The input is a horizontal force applied to the cart.

# Stabilisation using energy shaping: Problem description

- *Problem restatement:* Can we determine the set of potential functions that are achievable by using controls?

- If we only use control to alter the potential energy, it is possible to completely characterise the set of achievable potential functions. The set is often too small to be useful, e.g., for the pendulum/cart system, no stable potential is achievable in this way.

- *Question:* What if we allow not only the potential function to change, but also the kinetic energy metric?

- *Answer:* The set of achievable potential functions is then larger, e.g., for the pendulum/cart system there is now a stable potential achieved in this way.

- *Caveat:* To solve this problem requires solving a set of (generally overdetermined) nonlinear partial differential equations. . . gulp.

# Stabilisation using energy shaping: Methods

- Nonetheless, maybe we can answer the question of when a given system is stabilisable using this "energy shaping" strategy.

- Studying the partial differential equations is complicated. Here is a simple paradigm for understanding what is going on.
  *Problem:* In $\mathbb{R}^3$, given a vector field $X$, find a function $f$ so that $\operatorname{grad} f = X$.
  *Answer (from vector calculus):* There is a solution if and only if $\operatorname{curl} X = 0$.

- The condition $\operatorname{curl} X = 0$ is called a **compatibility condition**; it places the appropriate restrictions on the problem data to ensure that a solution exists.

- We have found the compatibility conditions for the energy shaping partial differential equations.

# Stabilisation using energy shaping: Methods

- This is really not trivial: it involves lots of Riemannian geometry and enough homological algebra to, for one thing, make sense of the following exact and commutative diagram

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & S^2(T^*M) & \xrightarrow{\sigma_1(\Phi)} & T^*M \otimes F & \xrightarrow{\tau} & K \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow \rho_1(R) & \longrightarrow & J^2(\mathbb{R}, E) & \xrightarrow{\rho_1(\Phi)} & J^1(\mathbb{R}, F) & & \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow R & \longrightarrow & J^1(\mathbb{R}, E) & \xrightarrow{\Phi} & F & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}
$$

which is used to construct the compatibility operator as a map from the bottom left corner to the top right corner.

# Summary

- Riemannian and affine differential geometry provide powerful tools for dealing with control problems for mechanical systems.