Tautological control systems

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What these ideas are and are not about

- They are not intended to be used to design control laws or, indeed, perform any other useful control theoretic tasks.
- The machinery in this talk is intended to provide a framework for studying *fundamental structural problems* in control theory, nothing more. . . but nothing less either.
- This talk is a mere sketch of the conference paper which is a mere sketch of a larger body of work:
  1. *Tautological Control Systems*, Springer-Verlag, 2014, 118pp+xii
  2. *Time-Varying Vector Fields and Their Flows* (with S. Jafarpour), Springer-Verlag, 2014, 119pp+viii

What is the “problem”?

- Why is a different framework needed from what is already out there? Let us consider the simplest possible illustration of this.
  - If one has a vector field X on a manifold M with an equilibrium point $x_0 \in$ M, the notions of “linearisation of X about x_0” and “linear stability of X at x_0” are unambiguous, i.e., understood in a coordinate-invariant way.
  - The same is not true of “linearisation of control systems,” “linear controllability of control systems,” and “linear stabilisability of control systems.”

What is the "problem"? (cont'd)

Example

Consider the two control-affine systems

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), & \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), & \dot{x}_3(t) &= u_2(t), \end{aligned}$$

- The systems are related by a simple feedback transformation and have the same trajectories.
- The system on the left has a linearisation that is neither controllable nor stabilisable and the linearisation on the right is controllable (and so stabilisable).

Conclusion: The notions of linearisation, linear controllability, and linear stabilisability are not feedback-invariant.

What is the “problem”? (cont’d)

- *Plea:* Do not try to “figure out” the example, but rather understand that it just says that the usual definitions have a lurking problem.
- As you know, nonlinear control theory is filled with many rather complicated constructions and theorems for doing things like determining when a system is controllable or stabilisable, and for determining the conditions for optimality of an extremal.
- Outside of the Lyapunov theory for stabilisation, there are likely very few constructions in nonlinear control theory that are feedback-invariant.
- To be able to address fundamental structural problems in control theory, one needs to have a feedback-invariant approach, or else hypotheses and/or conclusions will change with different system representations.

What is the “problem”? (cont’d)

- There are at least two approaches:
  - Make constructions with a given representation, and verify that these are, in fact, feedback-invariant.
  - Develop a methodology that is representation independent.
- The former is rather like making a coordinate construction in differential geometry and showing it, in fact, does not depend on the choice of coordinates, e.g., the linearisation of a vector field about an equilibrium point using the Jacobian in a set of coordinates.
- This approach seems really hard, probably impossible, definitely extremely messy.
- The latter approach is like making constructions in differential geometry that are *a priori* independent of coordinates.
- This latter approach is what we use here. It seems more elegant, but has its own difficulties.

Warning!

We are interested in “feedback-invariance,” not “feedback-invariants.”

Definitions

- Throughout, $\nu \in \{m, m+\text{lip}, \infty, \omega\}$ for $m \in \mathbb{Z}_{\geq 0}$ and $r = \omega$ if $\nu = \omega$ and $r = \infty$ otherwise.

Definition

A ***C$^\nu$-tautological control system*** is a pair $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ where

(i) M is a C^r-manifold and

(ii) $\mathscr{F}$ assigns to each open $\mathcal{U} \subseteq \mathsf{M}$ a subset $\mathscr{F}(\mathcal{U})$ of vector fields on $\mathcal{U}$ with the property that if $\mathcal{V} \subseteq \mathcal{U}$ then $X|\mathcal{V} \in \mathscr{F}(\mathcal{V})$ for every $X \in \mathscr{F}(\mathcal{U})$.

Definition

A C$^\nu$-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is ***globally generated*** if there exists a family $\mathscr{X}$ of globally defined vector fields such that

$$\mathscr{F}(\mathcal{U}) = \{X|\mathcal{U} \mid X \in \mathscr{X}\}.$$

Examples

Examples

1. An “ordinary” control system is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$ where M is the state manifold, $\mathcal{C}$ is the control set (assumed to be a topological space), and F is the dynamics:

$$\xi'(t) = F(\xi(t), \mu(t)),$$

$t \mapsto \mu(t) \in \mathcal{C}$ being the control and $t \mapsto \xi(t) \in \mathsf{M}$ being the trajectory. For each $u \in \mathcal{C}$ suppose that $F^u \colon x \mapsto F(x, u)$ is C^ν. Define a C^ν-tautological control system $\mathfrak{G}_\Sigma = (\mathsf{M}, \mathscr{F}_\Sigma)$ by

$$\mathscr{F}_\Sigma(\mathcal{U}) = \{F^u|\mathcal{U} \mid u \in \mathcal{C}\}.$$

Examples (cont'd)

Examples (cont'd)

2. Let $\mathsf{D} \subseteq \mathsf{TM}$ be a C^ν-distribution and define a C^ν-tautological control system $\mathfrak{G}_{\mathsf{D}} = (\mathsf{M}, \mathscr{F}_{\mathsf{D}})$ by

$$\mathscr{F}_{\mathsf{D}}(\mathcal{U}) = \{\mathsf{D}\text{-valued vector fields on } \mathcal{U} \text{ of class } \mathsf{C}^\nu\}.$$

This system is not globally generated.

3. Given a globally defined tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ define an "ordinary" control system $\Sigma_{\mathfrak{G}}$ with control set $\mathcal{C} = \mathscr{F}(\mathsf{M})$ and dynamics

$$\underbrace{F(x, X) = X(x)}_{\text{This is the tautology!}}$$

Correspondences (system)

- Note that we can go from a control system to a tautological control system back to a control system.
- Note that we can go from a globally defined tautological control system to a control system back to a tautological control system.

Proposition

Given a globally defined tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ *and a control system* $\Sigma = (\mathsf{M}, F, \mathcal{C})$*:*

(i) $\mathfrak{G}_{\Sigma_{\mathfrak{G}}} = \mathfrak{G}$*;*

(ii) $\Sigma_{\mathfrak{G}_{\Sigma}} = \Sigma$ *if the map* $u \mapsto F^u$ *is an homeomorphism onto its image.*

- We see here the first suggestion that topologies for spaces of vector fields are required in this framework. On this, more to come.

Trajectories

- In usual "family of vector fields" approaches, trajectories are concatenations of integral curves, i.e., piecewise constant controls.
- In the usual framework of $\dot{x} = F(x, u)$ one prescribes a control (say, bounded and measurable) to produce a time-varying vector field, and trajectories are integral curves of this vector field.
- We do not want to just do the first thing and we cannot do the second…
- Here's what we do:
  1. fix an open $\mathcal{U} \subseteq \mathsf{M}$ and interval $\mathbb{T} \subseteq \mathbb{R}$;
  2. let $\mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$ be the locally integrable mappings $X\colon \mathbb{T} \to \Gamma^\nu(\mathsf{T}\mathcal{U})$ such that $X(t) \in \mathscr{F}(\mathcal{U})$ for each $t \in \mathbb{T}$;
  3. a ***trajectory*** is an integral curve of some $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$.

Comments on topologies

- "Integrable" in the preceding slide means in the sense of the "Bochner integral."
- This requires a locally convex topology for $\Gamma^\nu(\mathrm{TM})$, which we have.
  - For $\nu \in \{m, \infty\}$ and (sort of) for $\nu = m + \text{lip}$, this is classical.
  - For $\nu = \omega$ this is new and nontrivial.
  - Explicit seminorms are given allowing us to explicitly characterise measurability and integrability.
  - We prove that, if $t \mapsto X_t$ is integrable, then the solutions of $\dot{x}(t) = X_t(x(t))$ exist, are unique, and depend on initial condition in a C^ν-manner.
  - For $\nu = \omega$, this is the only known result of this type.
- See Saber's talk later and our joint Springer booklet *Time-Varying Vector Fields and Their Flows*.

Correspondences (trajectories)

Theorem

If $\Sigma = (\mathsf{M}, F, \mathcal{C})$ *is an ordinary control system with* $\mathfrak{G}_\Sigma$ *the associated tautological control system, then:*

(i) *trajectories of* Σ *are trajectories of* $\mathfrak{G}_\Sigma$*;*

(ii) *if* $u \mapsto F^u$ *is continuous, injective, and proper, then trajectories of* $\mathfrak{G}_\Sigma$ *are trajectories of* Σ*;*

(iii) *if* $\mathcal{C}$ *is a Suslin space and if* F *is continuous and proper, then trajectories of* $\mathfrak{G}_\Sigma$ *are trajectories of* Σ.

Correspondences (trajectories) (cont'd)

Theorem

If $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is a globally generated tautological control system with $\Sigma_{\mathfrak{G}}$ the associated ordinary control system, then trajectories of $\mathfrak{G}$ and $\Sigma_{\mathfrak{G}}$ agree.

Corollary

(i) Trajectories of control-affine systems correspond to trajectories of the corresponding tautological control system.

(ii) If Σ is a control system with compact control set, trajectories of Σ correspond to trajectories of the corresponding tautological control system.

What's with the "sheaf" business?

- The definition of tautological control system includes the prescription of vector field families on all open sets.
- This idea comes from sheaf theory, and we will not say much about it here.
- One might feel that the most natural tautological control systems are those that are globally defined. This is false: *It is likely that assiduous attention to the sheaf theory aspects of tautological control theory will play an important part in the future development of the framework.*
- For example:
  1. sheaves systematise the notion of "germ" that is so important in local (in time and space) structure, and combined with our topologies give access to structure that is new;
  2. for flows, the analogue of sheaves for vector fields is groupoids;
  3. the map assigning to a time-varying vector field its flow becomes a homeomorphism in this framework.

What has been done?

- Apart from the basic constructions reported here:
  1. a study of transformations of tautological control systems;
  2. a theory of linearisation (harder than you might think);
  3. the Orbit Theorem for tautological control systems (S. Jafarpour);
  4. the beginning of optimal control theory (weirder than you might think, e.g., cost functions are sheaf morphisms).

What remains to be done?

Almost everything...