Why is Geometric Control Theory so Difficult?

Andrew D. Lewis

(Joint with Kaly Zhang)

Department of Mathematics and Statistics
Queen's University, Kingston, ON, Canada

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The importance of flows

- Control system:

$$\xi'(t) = F(t, \xi(t), \mu(t)),$$

  for a control $\mu\colon \mathbb{T} \to \mathcal{C}$ and trajectory $\xi\colon \mathbb{T} \to \mathsf{M}$.

- Flow: $(t, t_0, x_0) \mapsto \Phi^F(t, t_0, x_0, \mu)$.
- Reachable set:

$$\mathcal{R}_F(t, t_0, x_0) = \{\Phi^F(t, t_0, x_0, \mu) \mid \mu \in \mathscr{U}\},$$

  for some class of controls $\mathscr{U}$.

The importance of flows (cont'd)

- Three problems in control theory
  1. Controllability: some property of $\mathrm{int}(\mathcal{R}_F(t, t_0, x_0))$ for fixed (t_0, x_0).
  2. Stabilisability: there exists $\mu_* \in \mathscr{U}$ such that $\lim_{t\to\infty} \Phi^F(t, t_0, x, \mu_*) = x_0$ for all x.
  3. Optimality: find trajectories on the boundary of the reachable set for the cost-extended system.

Punchline

Understanding many control theoretic problems depends on the character of the reachable set, and the reachable set is characterised solely by flows.

Flows, naïvely

- What is a flow?
- Consider systems with time-domain $\mathbb{T}$ and state space M.
- Loosely, a flow assigns to an initial state $x_0 \in \mathsf{M}$, an initial time $t_0 \in \mathbb{T}$, and a final time $t \in \mathbb{T}$ a state $\Phi(t, t_0, x_0)$ which is where the state x_0 flows to in time $t - t_0$, starting at time t_0, i.e., one has a mapping

$$(t, t_0, x_0) \mapsto \Phi(t, t_0, x_0). \tag{1}$$

- This loose definition has serious defects:
  1. typically, flows are not defined for all $(t, t_0, x_0) \in \mathbb{T} \times \mathbb{T} \times \mathsf{M}$;
  2. what, exactly, are the properties of the mapping (1)?

Locality of flows

- For the first defect, consider

$$\xi'(t) = \mu(t)\xi(t)^2, \qquad t \in \mathbb{R},\ \xi(t) \in \mathbb{R},\ \mu(t) \in \mathbb{R}.$$

For every $\epsilon \in \mathbb{R}_{>0}$*,* $x_0 \in \mathbb{R} \setminus \{0\}$*, and* $t_0 \in \mathbb{R}$*, there exists* $\mu \in \mathsf{L}^1_{\mathrm{loc}}([t_0, \infty); \mathbb{R})$ *such that* $\Phi^F(t, t_0, x_0, \mu)$ *does not exist on* $[t_0, t_0 + \epsilon]$*.*

- However, flows *are* locally defined. Precisely, consider a time-varying, parameter-dependent vector field

$$X \colon \mathbb{T} \times \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$$

satisfying suitable conditions.[1] Given $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$, there exists an interval $\mathbb{T}' \subseteq \mathbb{T}$, $t_0 \in \mathbb{T}$, a neighbourhood $\mathcal{U}$ of x_0, and a neighbourhood $\mathcal{O}$ of p_0 such that Φ^X is defined on $\mathbb{T}' \times \mathbb{T}' \times \mathcal{U} \times \mathcal{O}$.

[1] cf. Jafarpour/L [1], *MCSS*, **28**(4), 1–46

Regularity of flows

- A time-varying, parameter-dependent flow should be taken as being defined on a flow-admissible[2] open subset $\mathcal{W} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M} \times \mathcal{P}$:

$$\Phi \colon \mathcal{W} \to \mathsf{M}.$$

- Regularity properties of flows
  1. Φ is continuous.
  2. Fix suitable $(t_0, x_0, p_0) \in \mathbb{T} \times \mathsf{M} \times \mathcal{P}$: $t \mapsto \Phi(t, t_0, x_0, p_0)$ is locally absolutely continuous.
  3. Fix suitable $(t, t_0, p_0) \in \mathbb{T} \times \mathbb{T} \times \mathcal{P}$: $x \mapsto \Phi(t, t_0, x, p_0)$ is of class C^ν for some regularity class ν, e.g., $\nu = \infty$ (smooth) or $\nu = \omega$ (real analytic).
- For a suitable time-varying, parameter dependent vector field X,[3,4] its flow Φ^X satisfies the above properties. (Including in the real analytic case!)

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[2]This means that, if $(t, t_0, x_0, p_0) \in \mathcal{W}$, then $[t, t_0] \times \{t_0\} \times \{x_0\} \times \{p_0\} \subseteq \mathcal{W}$.

[3]Jafarpour/L [1]

[4]Jafarpour/L [2] "Time-Varying Vector Fields and Their Flows," Springer

The “exponential map”

- For control systems, it is evident that the family of flows

$$(t, t_0, x_0) \mapsto \Phi^F(t, t_0, x_0, \mu), \qquad \mu \in \mathscr{U},$$

  is an entity of importance.
- This is typically thought of as the image of the family of time-varying vector fields

$$(t, x) \mapsto F(t, x, \mu(t)), \qquad \mu \in \mathscr{U},$$

  under some sort of “exponential map.”
- But, what *is* this exponential map? What is its domain? What is its codomain?
- Loosely, the domain is some subset of time-varying vector fields, and the codomain is some subset of flows. . .

“Families of diffeomorphisms”

- Given a time-varying vector field $X\colon \mathbb{T}\times \mathsf{M}\to \mathsf{TM}$, one has a “two-parameter family of diffeomorphisms”

$$(t,t_0)\mapsto (x\mapsto \Phi^X(t,t_0,x)).$$

- If $\mathsf{Diff}(\mathsf{M})$ is the group of diffeomorphisms of M, we have the induced family of mappings

$$\mathsf{Diff}(\mathsf{M})\ni\Psi\mapsto\Phi^X_{t,t_0}\circ\Psi\in\mathsf{Diff}(\mathsf{M}),\qquad (t,t_0)\in\mathbb{T}\times\mathbb{T}.$$

- Problem is. . . these are not actually diffeomorphisms; the problem of locality again.

Answer to question in title of talk

1. While we know what “the flow of a vector field” is, we do not know what “the *space* of flows” is.
2. While we “know” that a flow defines “a one-parameter family of diffeomorphisms,” we do not know what “the *space* of one-parameter family of *local* diffeomorphisms” is.
3. We do not have a good understanding of the maps in the sequence

$$\{\text{vector fields}\} \longrightarrow \{\text{local flows}\} \longrightarrow \{\text{local diffeomorphisms}\}$$

Spaces of flows

- We understand[5,6]

$$\{\text{vector fields}\} \longrightarrow \{\text{local flows}\} \longrightarrow \{\text{local diffeomorphisms}\}$$

- What about

$$\{\text{vector fields}\} \longrightarrow \{\text{local flows}\} \longrightarrow \{\text{local diffeomorphisms}\}$$

- Let $\mathbb{S}, \mathbb{S}' \subseteq \mathbb{T}$ be a subintervals, $\mathbb{S}' \supseteq \mathbb{S}$, and let $\mathcal{U} \subseteq \mathsf{M}$ be open. Let $\mathrm{LocFlow}^{\nu}(\mathbb{S}', \mathbb{S}, \mathcal{U})$ be those flows whose domain contains $\mathbb{S}' \times \mathbb{S} \times \mathcal{U}$.
- We can topologise the space $\mathrm{C}^{\nu}(\mathcal{U}; \mathsf{M})$ of local diffeomorphisms with domain $\mathcal{U} \subseteq \mathsf{M}$ as follows:
  1. topologise $\mathrm{C}^{\nu}(\mathcal{U})$ suitably;
  2. use the initial topology associated to the mappings

$$\begin{aligned} \psi_f \colon \mathrm{C}^{\nu}(\mathcal{U}; \mathsf{M}) &\to \mathrm{C}^{\nu}(\mathcal{U}) \\ \phi &\mapsto f \circ \phi, \end{aligned} \qquad f \in \mathrm{C}^{\nu}(\mathsf{M}).$$

[5] Jafarpour/L [2]

[6] Agrachev, Sachkov, "Control Theory from the Geometric Viewpoint," Springer

Spaces of flows (cont'd)

- We can topologise $\mathrm{LocFlow}^{\nu}(\mathbb{S}' \times \mathbb{S} \times \mathcal{U})$ as follows:
  1. give $\mathrm{C}^0(\mathbb{S}; \mathrm{C}^{\nu}(\mathcal{U}))$ the topology defined by the seminorms

$$p_{\mathbb{K}}(g) = \sup\{p \circ g(t_0) \mid t_0 \in \mathbb{K}\},$$

     $\mathbb{K} \subseteq \mathbb{S}$ a compact interval, p a seminorm for $\mathrm{C}^{\nu}(\mathcal{U})$;
  2. give $\mathrm{AC}(\mathbb{S}'; \mathrm{C}^0(\mathbb{S}; \mathrm{C}^{\nu}(\mathcal{U})))$ the topology defined by the seminorms

$$p^0_{\mathbb{K}'}(g) = \sup\{p_{\mathbb{K}} \circ g(t) \mid t \in \mathbb{K}'\}, \quad p^1_{\mathbb{K}'}(g) = \int_{\mathbb{K}'} p_{\mathbb{K}} \circ g'(t)\,\mathrm{d}t,$$

     $\mathbb{K} \subseteq \mathbb{S}$, $\mathbb{K}' \subseteq \mathbb{S}'$ compact intervals, p a seminorm for $\mathrm{C}^{\nu}(\mathcal{U})$;

Spaces of flows (cont'd)

3. note that a local flow $\Phi \in \mathsf{LocFlow}^{\nu}(\mathbb{S}' \times \mathbb{S} \times \mathcal{U})$ defines

$$\widehat{\Phi} \in \mathsf{AC}(\mathbb{S}'; \mathsf{C}^0(\mathbb{S}; \mathsf{C}^{\nu}(\mathcal{U}; \mathsf{M})))$$

   by $\widehat{\Phi}(t)(t_0)(x) = \Phi(t, t_0, x)$;
4. give $\mathsf{AC}(\mathbb{S}'; \mathsf{C}^0(\mathbb{S}; \mathsf{C}^{\nu}(\mathcal{U}; \mathsf{M})))$ the initial topology associated with the mappings

$$\begin{aligned} \psi_f \colon \mathsf{AC}(\mathbb{S}'; \mathsf{C}^0(\mathbb{S}; \mathsf{C}^{\nu}(\mathcal{U}; \mathsf{M}))) &\to \mathsf{AC}(\mathbb{S}'; \mathsf{C}^0(\mathbb{S}; \mathsf{C}^{\nu}(\mathcal{U}))) \\ \Psi &\mapsto f \circ \Psi, \end{aligned}$$

   for $f \in \mathsf{C}^{\nu}(\mathsf{M})$.

Spaces of flows (cont'd)

- For $\mathcal{W} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ be open and flow-admissible, and let $\mathrm{LocFlow}^{\nu}(\mathcal{W})$ be the set of flows whose domain contains $\mathcal{W}$.
- We can topologise $\mathrm{LocFlow}^{\nu}(\mathcal{W})$ as follows:
  1. cover $\mathcal{W}$ with a countable collection $C_j = \mathbb{S}'_j \times \mathbb{S}_j \times \mathcal{U}_j$, $j \in \mathbb{Z}_{>0}$, of flow-admissible cubes;
  2. consider the product $\prod_{j \in \mathbb{Z}_{>0}} \mathrm{LocFlow}^{\nu}(C_j)$;
  3. then $\mathrm{LocFlow}^{\nu}(\mathcal{W})$ is the subset of the product defined by requiring that flows agree on overlaps of cubes;
  4. use the subspace topology induced by the product topology;
  5. show that this topology is independent of the covering by cubes.

The exponential map

- The exponential map should be

$$\{\text{vector fields}\} \longrightarrow \{\text{local flows}\} \longrightarrow \{\text{local diffeomorphisms}\}$$

- We have just described $\{\text{local flows}\}$. What is $\{\text{vector fields}\}$?
- It is complicated, but if $\mathcal{W} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ is open and flow-admissible, then we can define and topologise

$$\mathscr{V}^{\nu}(\mathcal{W}) \subseteq \mathsf{L}^1_{\mathsf{loc}}(\mathbb{T}; \Gamma^{\nu}(\mathsf{TM})),$$

  the set of vector fields whose flows are defined on $\mathcal{W}$.
- Then $\exp\colon \mathscr{V}^{\nu}(\mathcal{W}) \to \mathsf{LocFlow}^{\nu}(\mathcal{W})$ is an homeomorphism.

Families of local diffeomorphisms

- As a byproduct, we have described

$$\{\text{vector fields}\} \longrightarrow \{\text{local flows}\} \longrightarrow \{\text{local diffeomorphisms}\}$$

- We now describe

$$\{\text{vector fields}\} \longrightarrow \{\text{local flows}\} \longrightarrow \{\text{local diffeomorphisms}\}$$

Families of local diffeomorphisms (cont'd)

- For an open and flow-admissible $\mathcal{W} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ and for subintervals $\mathbb{S} \subseteq \mathbb{S}' \subseteq \mathbb{T}$, denote

$$\mathcal{W}_{\mathbb{S}' \times \mathbb{S}} = \{x \in \mathsf{M} \mid \mathbb{S}' \times \mathbb{S} \times \{x\} \subseteq \mathcal{W}\}.$$

- If $\Phi \in \mathsf{LocFlow}^{\nu}(\mathcal{W})$, we have the two-parameter family of local diffeomorphisms

$$(t, t_0) \mapsto (x \mapsto \Phi(t, t_0, x)), \qquad x \in \mathcal{W}_{\mathbb{S}' \times \mathbb{S}}.$$

- If $\Psi \in \mathsf{C}^{\nu}(\mathcal{U}; \mathsf{M})$ is a local diffeomorphism with $\mathcal{U} \subseteq \mathcal{W}_{\mathbb{S}' \times \mathbb{S}}$, we have the induced family of homeomorphisms

$$\mathsf{C}^{\nu}(\mathcal{U}; \mathsf{M}) \ni \Psi \mapsto \Phi_{t,t_0} \circ \Psi \in \mathsf{C}^{\nu}(\mathcal{U}; \mathsf{M}), \qquad (t, t_0) \in \mathbb{S}' \times \mathbb{S}.$$

Control systems defined by flows

- For an open and flow-admissible $\mathcal{W} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$, we have a topological space $\mathrm{LocFlow}^{\nu}(\mathcal{W})$.
- We can define a type of control system by, for each open and flow-admissible subset $\mathcal{W} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$, assigning a subset $\mathscr{F}(\mathcal{W})$ of flows with the property that, if $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then

$$\{\Phi|\mathcal{W}_1 \mid \Phi \in \mathscr{F}(\mathcal{W}_2)\} \subseteq \mathscr{F}(\mathcal{W}_1).$$

- For a control system defined by flows, one can easily do all the control theoretic things one is used to, e.g., talk about controllability, optimality, and stabilisability.
- Now one formulates theorems using properties of flows and diffeomorphisms, rather than vector fields.
- In the case of flows defined by vector fields, one translates conditions on flows to conditions on vector fields by the homeomorphism of the exponential map.

Control systems defined by flows (cont'd)

- This is not a new idea![7]
- What *is* new is the careful inclusion in the formulation of the problem of locality.
- Here is an example of a global result about flows.[8]

Theorem

Let $\mathrm{Diff}_0(\mathsf{M})$ be the group of diffeomorphisms of a compact connected C^∞-manifold M and let $\mathscr{X} \subseteq \Gamma^\infty(\mathsf{TM})$ be bracket generating. Then

$$\{\Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1} \mid X_j \in \mathscr{X},\ t_j \in \mathbb{R}_{>0}\} = \mathrm{Diff}_0(\mathsf{M}).$$

- Relaxing compactness means bookkeeping the ensuing lack of completeness.

[7] Agrachev, Sachkov, Op. cit.

[8] Agrachev, Caponigro, *Ann. I. H. Poincaré – AN*, **26**(6), 2503–2509

Summary

- We have filled in all components of the cartoon

$$\{\text{vector fields}\} \longrightarrow \{\text{local flows}\} \longrightarrow \{\text{local diffeomorphisms}\}$$

- *And we have done this in a way that accounts for the problems of the locality of flows.*
- It remains to:
  1. prove theorems in terms of flows;
  2. translate those theorems into theorems about time-varying vector fields;
  3. translate those theorems into theorems about vector fields.