Composition, superposition, and ordinary differential equations

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Composition and superposition

- Suppose that one has spaces $\mathcal{X}$ and $\mathcal{Y}$ with structure giving rise to distinguished function (vector) spaces $\mathscr{F}_{\mathcal{X}}$ and $\mathscr{G}_{\mathcal{Y}}$. For example,
  1. $\mathcal{X}$ and $\mathcal{Y}$ could be topological spaces with $\mathscr{F}_{\mathcal{X}}$ and $\mathscr{G}_{\mathcal{Y}}$ the spaces of continuous functions,
  2. $\mathcal{X}$ and $\mathcal{Y}$ could be smooth, real analytic, or holomorphic manifolds with classes of functions $\mathscr{F}_{\mathcal{X}}$ and $\mathscr{G}_{\mathcal{Y}}$ of some regularity,
  3. $\mathcal{X}$ and $\mathcal{Y}$ could be measure spaces with $\mathscr{F}_{\mathcal{X}}$ and $\mathscr{G}_{\mathcal{Y}}$ some L^p classes of functions,
  4. ...
- Also suppose that there is a natural class of mappings $\mathscr{M}_{\mathcal{X},\mathcal{Y}}$ from $\mathcal{X}$ to $\mathcal{Y}$ that respect the structure of $\mathcal{X}$ and $\mathcal{Y}$. For example,
  1. continuous mappings,
  2. mappings with some regularity,
  3. measurable mappings,
  4. ...

Composition and superposition (cont'd)

- One now has a natural collection of operators.
  1. ***Composition operator:*** $$C_\Phi \colon \mathscr{G}_{\mathcal{Y}} \to \mathscr{F}_{\mathcal{X}}$$ $$g \mapsto g \circ \Phi$$ for fixed $\Phi \in \mathscr{M}_{\mathcal{X},\mathcal{Y}}$.
  2. ***Superposition operator:***[1] $$S_g \colon \mathscr{M}_{\mathcal{X},\mathcal{Y}} \to \mathscr{F}_{\mathcal{X}}$$ $$\Phi \mapsto g \circ \Phi$$ for fixed $g \in \mathscr{G}_{\mathcal{Y}}$.
  3. ***Joint composition operator:*** $$C_{\mathcal{X},\mathcal{Y}} \colon \mathscr{M}_{\mathcal{X},\mathcal{Y}} \times \mathscr{G}_{\mathcal{Y}} \to \mathscr{F}_{\mathcal{X}}$$ $$(\Phi, g) \mapsto g \circ \Phi.$$

[1] Also, ***Nemytskii operator*** or ***nonlinear composition operator***.

Composition and superposition (cont'd)

Questions

1. *Well-definedness:* For the chosen spaces of functions and mappings, are the operators well-defined?
   (The answer is sometimes provided by a classical theorem about composition.)
2. *Continuity:* Assuming that spaces of functions and mappings have topologies, are the operators continuous?
   (Answers can sometimes be surprising, as we shall see.)
3. *Boundedness:* Assuming that spaces of functions and mappings have bornologies, are the operators bornological?
   (For composition, boundedness and continuity are sometimes the same, but for superposition not necessarily so.)

- *Key fact:* Composition is typically linear, while superposition is typically nonlinear.

How can this have anything to do with ODEs?

Series representations of solutions to ODEs

- Let $r \in \{\infty, \omega\}$ and let M be a C^r-manifold.
- First consider the *time-independent* case.
  1. Let $X \in \Gamma^r(\mathsf{TM})$ and let $x_0 \in \mathsf{M}$.
  2. Think of $X \in \mathsf{L}(\mathrm{C}^r(\mathsf{M}); \mathrm{C}^r(\mathsf{M}))$ by $f \mapsto Xf \triangleq \mathscr{L}_X f$.
  3. For $t \in \mathbb{R}$ and $f \in \mathrm{C}^r(\mathsf{M})$, consider the *Volterra series* in $\mathsf{L}(\mathrm{C}^r(\mathsf{M}); \mathrm{C}^r(\mathsf{M}))$ given by

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{X \cdots X}_{k\text{-times}} f = \sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} X^k f \, \mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1 .$$

  4. If $r = \omega$, this converges for small t and for x near x_0 to the flow of X:

$$f \circ \Phi_t^X(x) = \left(\sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} X^k f \, \mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1 \right)(x).^2$$

[2] Agrachev, Gamkredlidze, *Math. USSR-Sb.*, **107**(4), 467–532, 1978

Series representations of solutions to ODEs (cont'd)

- Next consider the *time-dependent* case.
  1. Let $X \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}; \Gamma^r(\mathrm{TM}))$, let $t_0 \in \mathbb{R}$, and let $x_0 \in \mathrm{M}$.
  2. For fixed $t \in \mathbb{R}$, think of $X(t) \in \mathrm{L}(\mathrm{C}^r(\mathrm{M}); \mathrm{C}^r(\mathrm{M}))$ by $f \mapsto X(t)f \triangleq \mathscr{L}_{X(t)}f$.
  3. For $t \in \mathbb{R}$ and $f \in \mathrm{C}^r(\mathrm{M})$, the *Volterra series* in $\mathrm{L}(\mathrm{C}^r(\mathrm{M}); \mathrm{C}^r(\mathrm{M}))$ given by

$$\sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} X(t_1)X(t_2)\cdots X(t_k)f \, \mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1 .$$

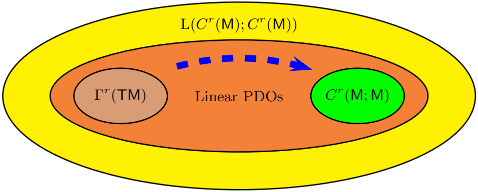
  4. If $r = \omega$, this converges for small t and for x near x_0 to the flow of X:

$$f \circ \Phi^X_t(x) = \left(\sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} X(t_1)X(t_2)\cdots X(t_k)f \, \mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1 \right)(x).$$ [3]

[3] Agrachev, Gamkredlidze, Op. cit.

Series representations of solutions to ODEs (cont'd)

- Drawbacks:
  1. doesn't converge for $r = \infty$;
  2. doesn't make sense for less than C^∞ regularity, e.g., the standard locally Lipschitz theory;
  3. one starts the series with a geometric entity in $L(C^r(M); C^r(M))$ (a vector field), and the limit is a geometric entity in $L(C^r(M); C^r(M))$ (a (local) diffeomorphism), but the intermediate partial sums are merely linear partial differential operators:



  4. One way to understand the talk: Can we remain in the green blob, and resolve some of the drawbacks?

Time- and parameter-dependent vector fields

- Typical hypotheses for right-hand sides of ODEs for existence, uniqueness, and continuous dependence of solutions look like this:[4]

To each point (α,β) in $\mathscr{F}$ there is a constant $\delta > 0$ and two integrable functions $M(t)$, $K(t)$ such that

1. The δ-neighborhood of (α,β) is in $\mathscr{F}$;
2. For each x in the δ-neighborhood β_δ of β and for each λ in Λ, the functions $f^i(t,x,\lambda)$ are measurable in t on the δ-neighborhood α_δ of α and satisfy the inequality

$$|f(t,x,\lambda)| \leq M(t) \tag{2.1}$$

on α_δ. Hence $f(t,x,\lambda)$ is integrable on $\alpha - \delta < t < \alpha + \delta$ for each x in β_δ and λ in Λ;

3. For each x and y in the δ-neighborhood β_δ of β and each λ in Λ, the inequality

$$|f(t,x,\lambda) - f(t,y,\lambda)| \leq K(t)\,|x - y| \tag{2.2}$$

holds on the δ-neighborhood α_δ of α;

4. For each x in β_δ and λ_0 in Λ we have

$$\lim_{\lambda=\lambda_0} \int_{\alpha-\delta}^{\alpha+\delta} |f(t,x,\lambda) - f(t,x,\lambda_0)|\, dt = 0. \tag{2.3}$$

[4] Hestenes, *Calculus of Variations and Optimal Control Theory*, Wiley, 1966

Time- and parameter-dependent vector fields (cont'd)

- If one wants more regular dependence on initial conditions, one needs more complicated conditions.
- We do this in a compact manner for a broad range of regularity classes and for sections of a C^r-vector bundle $\pi_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{M}$.
  1. Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\nu \in \mathbb{Z}_{\geq 0} \cup (\mathbb{Z}_{\geq 0} + \mathrm{lip}) \cup \{\infty, \omega, \mathrm{hol}\}$. Regularity "$m + \mathrm{lip}$" for $m \in \mathbb{Z}_{\geq 0}$ means "class C^m with locally Lipschitz top derivative."
  2. Locally convex topologies for $\Gamma^\nu(\mathsf{E})$ are classical, except for $\nu \in (\mathbb{Z}_{\geq 0} + \mathrm{lip}) \cup \{\omega\}$... but known in all cases...
  3. *Time-dependent* sections are in
  $$\boldsymbol{\Gamma^{\nu}_{\mathsf{LI}}(\mathbb{T}; \mathsf{E})} \triangleq \mathsf{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E}))(\simeq \mathsf{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}) \widehat{\otimes}_\pi \Gamma^\nu(\mathsf{E}))$$
  for $\mathbb{T} \subseteq \mathbb{R}$ an interval.
  4. *Time- and parameter-dependent* sections are in
  $$\boldsymbol{\Gamma^{\nu}_{\mathsf{PLI}}(\mathbb{T}; \mathsf{E}; \mathcal{P})} \triangleq \mathrm{C}^0(\mathcal{P}; \mathsf{L}^1_{\mathrm{loc}}(\mathbb{T}; \Gamma^\nu(\mathsf{E}))),$$
  for a topological space $\mathcal{P}$.

Time- and parameter-dependent mappings

- *Facts:*[5]
  1. a C^ν-vector field $X\colon \mathbb{T}\times M \to TM$ is in $\Gamma^\nu_{LI}(\mathbb{T};TM)$ if and only if $Xf \in C^\nu_{LI}(\mathbb{T};M)$ for every $f\in C^r(M)$;
  2. a C^ν-vector field $X\colon \mathbb{T}\times M\times \mathcal{P} \to TM$ is in $\Gamma^\nu_{PLI}(\mathbb{T};TM;\mathcal{P})$ if and only if $Xf\in C^\nu_{PLI}(\mathbb{T};M;\mathcal{P})$ for every $f\in C^r(M)$.
- *Definitions:* (locally absolutely continuous sections of $\pi_E\colon E\to M$)
  1. a C^ν-section $\xi\colon \mathbb{T}\times M\to E$ is in $\boldsymbol{\Gamma^\nu_{LAC}(\mathbb{T};E)}$ if

$$\xi(t)=\int_{t_0}^{t}\Xi(s)\,\mathrm{d}s$$

     for some $\Xi\in\Gamma^\nu_{LI}(\mathbb{T};E)$;
  2. $\boldsymbol{\Gamma^\nu_{PLAC}(\mathbb{T};E;\mathcal{P})} = C^0(\mathcal{P};\Gamma^\nu_{LAC}(\mathbb{T};E))$.
- *Definitions:* (locally absolutely continuous mappings)
  1. a C^ν-mapping $\Phi\colon\mathbb{T}\times M\to N$ is in $\boldsymbol{C^\nu_{LAC}(\mathbb{T};(M;N))}$ if $g\circ\Phi\in C^\nu_{LAC}(\mathbb{T};M)$ for every $g\in C^r(N)$;
  2. $\boldsymbol{C^\nu_{PLAC}(\mathbb{T};(M;N);\mathcal{P})} = C^0(\mathcal{P};C^\nu_{LAC}(\mathbb{T};(M;N)))$.

[5] Technical point: For $\nu=\mathrm{hol}$, these statements require M to be a "Stein manifold." We will just assume this to always be the case.

Properties of $\mathrm{C}^{\mathrm{lip}}_{\mathrm{PLAC}}(\mathbb{T};(\mathsf{M};\mathsf{N});\mathcal{P})$

- Of course, lip-regularity is classically important, and so interesting to think about.
  1. $\mathrm{C}^{\mathrm{lip}}_{\mathrm{PLAC}}(\mathbb{T};(\mathsf{M};\mathsf{N});\mathcal{P}) \subseteq \mathrm{C}^0(\mathbb{T}\times\mathsf{M}\times\mathcal{P};\mathsf{N})$ (actually only requires $\nu=0$).
  2. $t\mapsto\Phi^p_x(t)=\Phi(t,x,p)$ are locally absolutely continuous (again, only requires $\nu=0$).
  3. $\lim_{(x,p)\to(x_0,p_0)}\Phi^p_x=\Phi^{p_0}_{x_0}$, uniformly on compact subintervals of $\mathbb{T}$.
  4. Let
     1. $K\subseteq\mathsf{M}$ and $\mathbb{K}\subseteq\mathbb{T}$ be compact and
     2. $p_0\in\mathcal{P}$,

     denote

     3. $K_0=\cup_{(t,x)\in\mathbb{K}\times K}\Phi(t,x,p_0)$,

     and let

     4. $\mathcal{V}$ be a neighbourhood of K_0.

     Then there exists a neighbourhood $\mathcal{O}$ of p_0 such that

$$\bigcup_{(t,x,p)\in\mathbb{K}\times K\times\mathcal{O}}\Phi(t,x,p)\subseteq\mathcal{V}.$$

The main idea

- One fantasises that, given $X \in \Gamma^{\nu}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{TM}; \mathcal{P})$, its flow (from a fixed initial time t_0) is in $\mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{M}); \mathcal{P})$.
- The fantasy is false in an obvious way: the flow will generally only be defined on a subset of $\mathbb{T} \times \mathsf{M} \times \mathcal{P}$, i.e., flows are only *local* flows. Let us sidestep this issue for the moment...
- We can try to realise our fantasy by a procedure rather like Picard iteration in the classical setting.
  1. Take $\Phi^{p}_{0,t}(x) = \Phi_0(t, x, p) = x$, i.e., $\Phi^{p}_{0,t} = \mathrm{id}_{\mathsf{M}}$.
  2. Given $\Phi_k \colon \mathbb{T} \times \mathsf{M} \times \mathcal{P} \to \mathsf{M}$, define Φ_{k+1} by

$$g \circ \Phi^{p}_{k+1,t} = g + \int_{t_0}^{t} X^{p}_{s} g \circ \Phi^{p}_{k,s} X^{p}_{s} g \circ \Phi^{p}_{k,s} \, \mathrm{d}s, \qquad g \in \mathrm{C}^{r}(\mathsf{M}).$$

  3. We have seen that $(s, x, p) \mapsto X^{p}_{s} g \in \mathrm{C}^{\nu}_{\mathrm{PLI}}(\mathbb{T}; \mathsf{M}; \mathcal{P})$.
  4. The hard part is handling the time- and parameter-dependent *superposition* operation $\Phi^{p}_{s} \mapsto X^{p}_{s} g \circ \Phi^{p}_{s}$ for $\Phi \in \mathrm{C}^{\nu}_{\mathrm{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{M}); \mathcal{P})$.

A theorem on time-dependent superposition

Theorem

Let $r \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$*. Let* M *and* N *be* C^r*-manifolds. If* $\Phi \in \mathrm{C}^{\nu}_{\mathrm{LAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}))$ *and if* $g \in \mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{N})$*, then the mapping*

$$t \mapsto g_t \circ \Phi_t$$

defines an element of $\mathrm{C}^{\nu}_{\mathrm{LI}}(\mathbb{T}; \mathsf{M})$*.*

A theorem on time- and parameter-dependent superposition

Theorem

Let $r \in \{\infty, \omega, \mathrm{hol}\}$*, let* $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$*, and denote*

$$\nu' = \begin{cases} m + \mathrm{lip}, & m \in \mathbb{Z}_{\geq 0}, \\ \nu, & \nu \in \{\infty, \omega, \mathrm{hol}\}. \end{cases}$$

Let M *and* N *be* C^r*-manifolds and let* $\mathcal{P}$ *be a topological space. If* $\Phi \in \mathsf{C}^{\nu}_{\mathsf{PLAC}}(\mathbb{T}; (\mathsf{M}; \mathsf{N}); \mathcal{P})$ *and if* $g \in \mathsf{C}^{\nu'}_{\mathsf{PLI}}(\mathbb{T}; \mathsf{N}; \mathcal{P})$*, then the mapping*

$$\mathcal{P} \ni p \mapsto \left(t \mapsto \int_{t_0}^{t} g^p_s \circ \Phi^p_s \, \mathrm{d}s \right)$$

defines an element of $\mathsf{C}^{\nu}_{\mathsf{PLAC}}(\mathbb{T}; \mathsf{M}; \mathcal{P})$*.*

A theorem on time- and parameter-dependent superposition (cont'd)

Remarks

1. Part of the proof of the theorem is proving that the joint composition map

$$\begin{aligned} C_{\mathsf{M},\mathsf{N}} \colon \mathrm{C}^{\nu}(\mathsf{M};\mathsf{N}) \times \mathrm{C}^{\nu}(\mathsf{N}) &\to \mathrm{C}^{\nu}(\mathsf{M}) \\ (\Phi, g) &\mapsto g \circ \Phi \end{aligned}$$

   is continuous.
   a. This is classical for $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \mathrm{hol}\}$.[a]
   b. This is true but not classical for $\nu = \omega$.[b]
   c. This is false for $\nu \in \mathbb{Z}_{\geq 0} + \mathrm{lip}$; the superposition operator S_g is continuous for $\nu = \mathrm{lip}$ if and only if $g \in \mathrm{C}^1(\mathsf{N})$.[c]

[a] Exercise 2.4.10 in Hirsch's *Differential Topology*, Springer, 1976
[b] L, *Geometric analysis on real analytic manifolds*, Springer, 2023
[c] Drábek, *Comment. Math. Univ. Carolin.*, **16**(1), 37-57, 1975

A theorem on time- and parameter-dependent superposition (cont'd)

Remarks

2. Because of the lack of continuity of the superposition operator for $\nu \in \mathbb{Z}_{\geq 0} + \text{lip}$, the previous theorem does not hold for such ν.
3. For the theorem to hold for $\nu \in \mathbb{Z}_{\geq 0}$, we require that g have regularity $\nu' = \nu + \text{lip}$.
4. *Punchline:* Getting the continuous dependence on the parameter in our Picard-like iteration is subtle: to get C^m-regular parameter dependence of Φ_{k+1}, one must assume that X has regularity $\mathrm{C}^{m+\text{lip}}$.
5. Our iterative method does not require infinite differentiability since successive iterates involve composition, not differentiation.
6. One starts the iteration with a mapping, at each iteration step one defines a new mapping, and convergence is to a mapping.
7. *Trade off:* superposition is nonlinear!

Convergence of iteration scheme

- Some cases are done, but convergence is a work in progress...
- *Acknowledgements:* Saber Jafarpour and Kaly Zhang

The End