Variational and nonholonomic mechanics

or “How I haven’t done anything since I was Richard’s PhD student”

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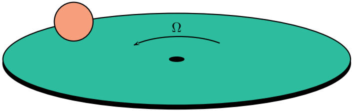
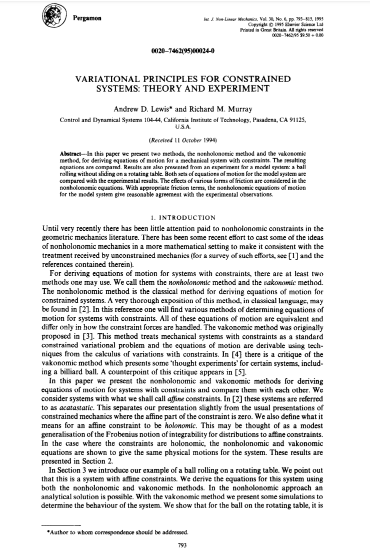
~~UCSB Gathering of Elderly Researchers~~
Vistas in Control
20/10/2023

Genesis



- Studied two ways of modelling the system with nonholonomic constraints
- Did some friction modelling
- Compared numerics and data collected from a sophisticated experiment apparatus[a]

[a] A stereo turntable, a ping-pong ball, and a VHS camera



Pergamon

Int. J. Non-Linear Mechanics, Vol. 30, No. 6, pp. 793–815, 1995

0020–7462/95 $9.50 + 0.00

0020–7462(95)00024-0

VARIATIONAL PRINCIPLES FOR CONSTRAINED SYSTEMS: THEORY AND EXPERIMENT

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(*Received* 11 *October* 1994)

Abstract—In this paper we present two methods, the nonholonomic method and the vakonomic method, for deriving equations of motion for a mechanical system with constraints. The resulting equations are compared. Results are also presented from an experiment for a model system: a ball rolling without sliding on a rotating table. Both sets of equations of motion for the model system are compared with the experimental results. The effects of various forms of friction are considered in the nonholonomic equations. With appropriate friction terms, the nonholonomic equations of motion for the model system give reasonable agreement with the experimental observations.

1. INTRODUCTION

Until very recently there has been little attention paid to nonholonomic constraints in the geometric mechanics literature. There has been some recent effort to cast some of the ideas of nonholonomic mechanics in a more mathematical setting to make it consistent with the treatment received by unconstrained mechanics (for a survey of such efforts, see [1] and the references contained therein).

For deriving equations of motion for systems with constraints, there are at least two methods one may use. We call them the *nonholonomic* method and the *vakonomic* method. The nonholonomic method is the classical method for deriving equations of motion for constrained systems. A very thorough exposition of this method, in classical language, may be found in [2]. In this reference one will find various methods of determining equations of motion for systems with constraints. All of these equations of motion are equivalent and differ only in how the constraint forces are handled. The vakonomic method was originally proposed in [3]. This method treats mechanical systems with constraints as a standard constrained variational problem and the equations of motion are derivable using techniques from the calculus of variations with constraints. In [4] there is a critique of the vakonomic method which presents some 'thought experiments' for certain systems, including a billiard ball. A counterpoint of this critique appears in [5].

In this paper we present the nonholonomic and vakonomic methods for deriving equations of motion for systems with constraints and compare them with each other. We consider systems with what we shall call *affine* constraints. In [2] these systems are referred to as *acatastatic*. This separates our presentation slightly from the usual presentations of constrained mechanics where the affine part of the constraint is zero. We also define what it means for an affine constraint to be *holonomic*. This may be thought of as a modest generalisation of the Frobenius notion of integrability for distributions to affine constraints. In the case where the constraints are holonomic, the nonholonomic and vakonomic equations are shown to give the same physical motions for the system. These results are presented in Section 2.

In Section 3 we introduce our example of a ball rolling on a rotating table. We point out that this is a system with affine constraints. We derive the equations for this system using both the nonholonomic and vakonomic methods. In the nonholonomic approach an analytical solution is possible. With the vakonomic method we present some simulations to determine the behaviour of the system. We show that for the ball on the rotating table, it is

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Time passes. . .

family. . .

boating. . .

reading. . .

relaxing. . .

Revelation

- After 25 years…back to the drawing board…
- Some other contributions in the intervening years:
  1. Cardin/Favretti, *J. Geom. Phys.*, **18**(4), 295–325, 1996
  2. Favretti, *J. Dyn. Diff. Eq.*, **10**(4), 511-536, 1998
  3. Zampieri, *J. Diff. Eq.*, **163**(2), 335-347, 2000
  4. Kupka/Oliva, *J. Diff. Eq.*, **169**(1), 169–189, 2001
  5. Cortés/de Léon/Martín de Diego/Martínez, *SIAM J. Control Optim.*, **41**(5), 1389–1412, 2002
  6. Fernandez/Bloch, *J. Phys. A*, **41**(3), no. 344005, 2008
  7. Terra, *São Paulo J. Math. Sci.*, **12**(1), 136-145, 2018
  8. Jóźwikowski/Respondek, *J. Geom. Mech.*, **11**(1), 77-122, 2019

JOURNAL OF GEOMETRIC MECHANICS
doi:10.3934/jgm.2020013

NONHOLONOMIC AND CONSTRAINED VARIATIONAL MECHANICS

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(Communicated by Witold Respondek)

Abstract. Equations governing mechanical systems with nonholonomic constraints can be developed in two ways: (1) using the physical principles of Newtonian mechanics; (2) using a constrained variational principle. Generally, the two sets of resulting equations are not equivalent. While mechanics arises from the first of these methods, sub-Riemannian geometry is a special case of the second. Thus both sets of equations are of independent interest.

The equations in both cases are carefully derived using a novel Sobolev analysis where infinite-dimensional Hilbert manifolds are replaced with infinite-dimensional Hilbert spaces for the purposes of analysis. A useful representation of these equations is given using the so-called constrained connection derived from the system's Riemannian metric, and the constraint distribution and its orthogonal complement. In the special case of sub-Riemannian geometry, some observations are made about the affine connection formulation of the equations for extremals.

Using the affine connection formulation of the equations, the physical and variational equations are compared and conditions are given that characterise when all physical solutions arise as extremals in the variational formulation. The characterisation is complete in the real analytic case, while in the smooth case a locally constant rank assumption must be made. The main construction is that of the largest affine subbundle variety of a subbundle that is invariant under the flow of an affine vector field on the total space of a vector bundle.

Contents

2010 *Mathematics Subject Classification.* Primary: 70F25; Secondary: 34C45, 46E35, 49K05, 53B05, 53C05, 53C17, 58A30, 70G45.

Key words and phrases. Nonholonomic mechanics, calculus of variations, affine differential geometry, Sobolev spaces of mappings, linear and affine vector fields on vector bundles, invariant subbundles, sub-Riemannian geometry.

Research supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

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Beware

Listen more than see

I will often write precise mathematical statements without defining the notation. Notation will be explained during the talking part of the talk.

Two problems

- Data:
  1. configuration manifold Q of regularity $r \in \{\infty, \omega\}$;
  2. kinetic energy modelled by a C^r-Riemannian metric $\mathbb{G}$;
  3. potential energy modelled by a potential function $V \in C^r(Q)$;
  4. nonholonomic constraints modelled by a C^r-distribution $D \subseteq TQ$.
- Action for a Lagrangian $L\colon TQ \to \mathbb{R}$ is

$$A_L(\gamma) = \int_{t_0}^{t_1} L \circ \gamma'(t)\,\mathrm{d}t,$$

for $\gamma \in H^1([t_0, t_1]; Q; q_0, q_1)$.

Problem (Nonholonomic (N))

Find $\gamma \in H^1([t_0, t_1]; Q; D; q_0, q_1)$ *such that*

$$\langle \mathrm{d}(A_{\mathbb{G}} - A_V); \delta \rangle = 0,$$

$$\delta \in H^1([t_0, t_1]; \gamma^* D; q_0, q_1).$$

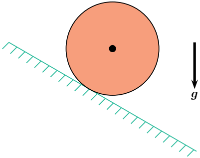
Problem (Variational (V))

Find $\gamma \in H^1([t_0, t_1]; Q; D; q_0, q_1)$ *such that*

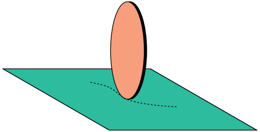
$$\langle \mathrm{d}(A_{\mathbb{G},D} - A_{V,D}); \delta\sigma(0) \rangle = 0,$$

$$\sigma\colon (-\epsilon, \epsilon) \to H^1([t_0, t_1]; Q; D; q_0, q_1).$$

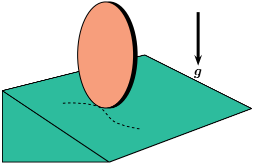
Three examples



All solutions to Problems (N) and (V) give the same physical motions.



For every solution to Problem (N), *there exists* a solution to Problem (V) giving the same physical motion.



For *almost no* solution to Problem (N) does there exist a solution to Problem (V) that gives the same physical motion.[a]

[a] Lemos, *Acta Mech.*, **233**, 47-56, 2022

Problem Statement

Problem (Vague version)

Characterise the set of initial conditions for which the solution to Problems (N) and (V) gives the same physical motion.

- To make this vague statement more precise, one needs to understand more about the solutions to Problems (N) and (V).

Some connections and tensors

Given a Riemannian manifold $(\mathsf{Q}, \mathbb{G})$ and a distribution D:

1. ***Levi-Civita connection:*** $\nabla^{\mathbb{G}}$
2. $\mathbb{G}$-orthogonal projections P_{D} and $P_{\mathsf{D}^\perp}$
3. ***constrained connection:*** ∇^{D} (project $\nabla^{\mathbb{G}}$ onto D)
4. ***Fröbenius curvature:*** $F_{\mathsf{D}}(X, Y) = P_{\mathsf{D}^\perp}([X, Y]) \quad (X, Y \in \Gamma^r(\mathsf{D}))$
5. ***geodesic curvature:*** $G_{\mathsf{D}}(X, Y) = P_{\mathsf{D}^\perp}(\nabla^{\mathbb{G}}_X Y + \nabla^{\mathbb{G}}_Y X) \quad (X, Y \in \Gamma^r(\mathsf{D}))$

Solutions to Problem (N)

Theorem

For $\gamma \in \mathsf{H}^1([t_0, t_1]; \mathsf{Q}; \mathsf{D}; q_0, q_1)$*, the following statements are equivalent:*

1. γ *is a solution to Problem (N);*
2. $\gamma \in \mathsf{H}^2([t_0, t_1]; \mathsf{Q})$ *and there exists* $\lambda \in \mathsf{L}^2([t_0, t_1]; \gamma^* \mathsf{D}^\perp)$ *such that*
$$\nabla^{\mathbb{G}}_{\gamma'} \gamma' + \operatorname{grad} V \circ \gamma = \lambda;$$
3. $\gamma \in \mathsf{H}^2([t_0, t_1]; \mathsf{Q})$ *and satisfies*
$$\nabla^{\mathsf{D}}_{\gamma'} \gamma' + P_{\mathsf{D}} \circ \operatorname{grad} V \circ \gamma = 0.$$

Solutions to Problem (V)

Theorem

For $\gamma \in \mathsf{H}^1([t_0,t_1];\mathsf{Q};\mathsf{D};q_0,q_1)$, *the following statements are equivalent:*

1. γ *is a solution to Problem (V);*
2. *at least one of the following holds:*
   - a. *some interesting condition for singular extremals that I will ignore, sacrificing correctness for expediency;*
   - b. $\gamma \in \mathsf{H}^2([t_0,t_1];\mathsf{Q})$ *and there exists* $\lambda \in \mathsf{H}^1([t_0,t_1];\gamma^*\mathsf{D}^\perp)$ *such that*

$$\nabla^{\mathbb{G}}_{\gamma'}\gamma' + \operatorname{grad} V \circ \gamma - \nabla^{\mathbb{G}}_{\gamma'}\lambda - S^*_{\mathsf{D}}(\gamma')(\lambda) = 0;$$ [a]

3. *at least one of the following holds:*
   - a. *some other interesting condition for singular extremals that I will again ignore;*
   - b. $\gamma \in \mathsf{H}^2([t_0,t_1];\mathsf{Q})$ *and there exists* $\lambda \in \mathsf{H}^1([t_0,t_1];\gamma^*\mathsf{D}^\perp)$ *such that*

$$\nabla^{\mathsf{D}}_{\gamma'}\gamma' + P_{\mathsf{D}} \circ \operatorname{grad} V \circ \gamma = F^*_{\mathsf{D}}(\gamma')(\lambda),$$
$$\nabla^{\mathsf{D}^\perp}_{\gamma'}\lambda = \frac{1}{2}G_{\mathsf{D}}(\gamma',\gamma') + P_{\mathsf{D}^\perp} \circ \operatorname{grad} V \circ \gamma + \frac{1}{2}G^\star_{\mathsf{D}^\perp}(\gamma')(\lambda) + \frac{1}{2}F^\star_{\mathsf{D}^\perp}(\gamma')(\lambda).$$

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[a] Kupka/Oliva, *J. Diff. Equations*, **169**(1), 169–189, 2001

The crucial observation

Compare

$$\nabla^{\mathsf{D}}_{\gamma'}\gamma' + P_{\mathsf{D}} \circ \operatorname{grad} V \circ \gamma = 0$$

with

$$\begin{aligned}
&\nabla^{\mathsf{D}}_{\gamma'}\gamma' + P_{\mathsf{D}} \circ \operatorname{grad} V \circ \gamma = F^*_{\mathsf{D}}(\gamma')(\lambda), \\
&\nabla^{\mathsf{D}^\perp}_{\gamma'}\lambda = \frac{1}{2}G_{\mathsf{D}}(\gamma',\gamma') + P_{\mathsf{D}^\perp} \circ \operatorname{grad} V \circ \gamma + \frac{1}{2}G^\star_{\mathsf{D}^\perp}(\gamma')(\lambda) + \frac{1}{2}F^\star_{\mathsf{D}^\perp}(\gamma')(\lambda).
\end{aligned} \tag{1}$$

Problem

Given a physical motion $t \mapsto \gamma(t)$ satisfying Problem (N), find all (if any) initial conditions for λ so that the resulting solution to (1) *is such that $F^*_{\mathsf{D}}(\gamma')(\lambda) = 0$.*

Symbolic abstraction

If we think of γ as given, the equation (1) for λ has the form of an affine differential equation,

$$\underbrace{\nabla^{\mathsf{D}^\perp}_{\gamma'}\lambda}_{\dot{\lambda}(t)} = \underbrace{\frac{1}{2}G^\star_{\mathsf{D}^\perp}(\gamma')(\lambda) + \frac{1}{2}F^\star_{\mathsf{D}^\perp}(\gamma')(\lambda)}_{A(t)(\lambda(t))} + \underbrace{\frac{1}{2}G_{\mathsf{D}}(\gamma',\gamma') + P_{\mathsf{D}^\perp}\circ \operatorname{grad} V\circ\gamma}_{b(t)},$$

and the condition satisfied by λ is a “satisfies a linear equation” condition,

$$\underbrace{F^*_{\mathsf{D}}(\gamma')(\lambda) = 0}_{B(t)(\lambda(t))=0}.$$

Problem (In symbols)

Find all solutions of the affine differential equation

$$\dot{\lambda}(t) = A(t)(\lambda(t)) + b(t)$$

satisfying $B(t)(\lambda(t)) = 0$.

Complete abstraction

We have the following data:

1. a C^r-vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ (abstracting the pull-back bundle $\pi_{\mathsf{D}}^*\mathsf{D}^\perp \to \mathsf{D}$);
2. a C^r-cogeneralised subbundle $\mathsf{F} \subseteq \mathsf{E}$ (abstracting $\ker(B)$);
3. an affine vector field X on E (abstracting $\dot{\lambda} = A \circ \lambda + b$) over a vector field X_0 on M (abstracting the equations governing Problem (N)).

Problem (WDSo(N)a(V)A)

Find the subset A *of initial conditions in* E *through which integral curves of* X *remain in* F*.*

Final piece of the jigsaw: just elementary linear algebra

Proposition (Linear equations)

Let V *be a finite-dimensional* $\mathbb{R}$*-vector space. There is a 1–1 correspondence between the sets of solutions of linear equations*

$$A(v) + b = 0, \qquad A \in \mathsf{End}_{\mathbb{R}}(\mathsf{V}),\ b \in \mathsf{V},$$

and subspaces $\Delta \subseteq \mathsf{V}^* \oplus \mathbb{R}$ *with positive codimension. Moreover, the set of solutions to the linear equation is nonempty if and only if* $(0, 1) \in \Delta$*.*

- Making this geometric, for a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, we think of subbundles of positive codimension of $\mathsf{E}^* \oplus \mathbb{R}_{\mathsf{M}}$ as being bundles of linear equations.
- The affine vector field X induces a *linear* vector field in this bundle of linear equations.

Results

Theorem

Assume X_0 is complete.
The solution A *of Problem (WDSo(N)a(V)A) is the largest generalised linear equation subbundle of* $\mathsf{E}^* \oplus \mathbb{R}_{\mathsf{M}}$ *whose solutions are a cogeneralised affine subbundle* A *over a variety* $\mathsf{S} \subseteq \mathsf{M}$*. Also, this exists.*

Once more, in English:

The set of initial conditions for λ that give rise to solutions of Problem (V) giving physical motions that are solutions to Problem (N) are described by

1. a (possibly empty) subvariety $\mathsf{S} \subseteq \mathsf{M}$ (think submanifold, if you want) and,
2. for each $x \in \mathsf{S}$, an affine subspace $\mathsf{A}_x \subseteq \mathsf{F}_x$.

There's more…infinitesimal conditions satisfied by "A"…a PDE one can analyse à la Spencer…connections to sub-Riemannian geometry…but…

...I have important things to do...