A Mathematical Approach to Classical Control

Single-input, single-output, time-invariant, continuous time, finite-dimensional, deterministic, linear systems

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Preface

The purpose of this preface is two-fold: (1) to discuss the philosophy of the approach taken, as it is nonstandard for an introductory course; (2) to discuss the content of the book.

The philosophy

Since this book takes an untraditional approach to introductory control, it is worth outlining why I have taken the approach I have.

The goals Clearly a new text in classical control needs to have some justification for its appearance, as there are already a large number of texts on the market, and these satisfy the demands made by a typical introductory course in feedback control. The approach in this book is not typical. The idea here is to develop control theory, at an introductory classical level, as a rigorous subject. This is different, note, from presenting the mathematics needed to understand control theory. One will often hear things like, “Classical control is merely an application of complex variable theory,” or “Linear control is merely an application of linear algebra.” While it is true that these parts of control theory do rely on the asserted branches of mathematics, control theory is such an effective blend of many branches of mathematics that to categorise it as a subset of one is a disservice. The subject of control theory, even at an introductory level, has a mathematical life of its own, and it is this life that is being exhibited here.

The main benefit of such an approach is that not just the mathematics behind the subject, but the subject itself can be treated rigorously. The problems of control theory, and these are actual practical problems, often have precise mathematical statements, and the intent in this book is to give these wherever possible. The result is that a student will be able to understand simple problems in a larger context. For some, at least, this is useful. It also makes it possible to consider challenging control problems that cannot really be considered in an exclusively ad hoc treatment. It would seem that many classical control texts were written based upon the standard of control practice in, say, the early 1960’s. This practice, well laid out in the texts of Truxal [1955] and Horowitz [1963], had reached a point where, for the problems to which it was applicable, it was “finished.” This was expressed in one early paper as follows: “The present state of the art is such that it is safe to assume that, for linear single-loop feedback systems, almost no analysis or design problems of any consequence remain.” Such statements are seldom prophetic. Indeed, much has been done since the date of publication of the cited paper (1961), even for linear single-loop systems. Now we have means for handling problems that would be almost impossible to treat using the ad hoc methods of classical design. And the methods all rely on a firm grasp of not just the mathematics behind control theory, but the mathematics of the subject itself. This is the reason for this book.

The mathematical approach With the above as backdrop, this book is provided for students who can be relied upon to have a satisfactory background in linear algebra, differential equations (including the matrix exponential), basic complex analysis, and some transform theory. The appendices contain a quick overview of necessary background material, so that an instructor or a student can determine whether the book is useful.

Apart from the above pedagogical concerns, I have also tried to write the book with an eye towards its being a useful reference. In the book, I have tried to prove as many statements as possible; even many that are not commonly proved, but often stated. I do this not because I feel that all of these proofs should be delivered in lectures—I certainly do not do this myself. Rather, my objectives here are scholarly. I do not feel that such lofty goals clash with the rather more pedantic concerns of getting students to come to grips with basic material. Students who find the course challenging may safely omit consideration of the more technical proofs, provided that they understand the concepts behind the results. More curious students, however, are rewarded by having for reference proofs that can be difficult to find in the literature. Moreover, this approach has, in my experience, a pedagogical byproduct. If one teaches an introductory course in a manner not completely “method oriented,” natural questions will arise in the presentation. For example, if one even gets around to posing the problem of finding a controller that stabilises a given plant in a unity gain feedback loop, the natural question arises as to whether such controllers exist. The answer is affirmative, but the determination of this answer is nontrivial. A traditional approach to classical control masks the existence of the question, never mind providing the answer. Again, the advantage of the approach taken here, at least for the curious student, is that the answer to this more basic question may be found alongside the more standard ad hoc methods for controller design.

The rôle of control design A word needs to be said about control design. Greater emphasis is being placed on engineering design in the engineering undergraduate curriculum, and this is by all means an appropriate tendency. When teaching a control course, one faces a decision relative to design content. Should the design be integrated into the course at every stage, or should it be separated from the analysis parts of the course? In this book, the trend in engineering education is being bucked, and the latter course is taken. Indeed, care has been taken to explicitly separate the book into three parts, with the design part coming last. One can justly argue that this is a mistake, but it is the approach I have decided upon, and it seems to work. My rationale for adopting the approach I do is that in control, there is very simply a lot of analysis to learn before one can do design in a fulfilling way. Thus I get all the tools in place before design is undertaken in the latter stages of the book.

How to use the book It is not possible to cover all of the topics in this book in a single term; at least it is not advisable to attempt this. However, it is quite easy to break the contents of the book into two courses, one at an introductory level, and another dealing with advanced topics. Because this division is not readily made on a chapter-by-chapter basis, it is perhaps worth suggesting two possible courses that can be taught from this book.

An introductory course for students with no control background might contain roughly the following material:

1. Chapter 1;
2. Chapter 2, possibly omitting details about zero dynamics (Section 2.3.3), and going lightly on some of the proofs in Section 2.3;
3. Chapter 3, certainly going lightly on the proofs in Section 3.3;
4. Chapter 4, probably omitting Bode’s Gain/Phase Theorem (Section 4.4.2) and perhaps material about plant uncertainty models (Section 4.5).
5. Chapter 5, omitting many of the details of signal and system norms in Section 5.3, omitting Liapunov stability (Section 5.4), and omitting the proofs of the Routh/Hurwitz criteria;
6. Chapter 6, going lightly, perhaps, on the detailed account of signal flow graphs in Sections 6.1 and 6.2, and covering as much of the material in Section 6.4 as deemed appropriate; the material in Section 6.5 may form the core of the discussion about feedback in a more traditional course;
7. Chapter 7, probably omitting robust stability (Section 7.3);
8. Chapter 8;
9. maybe some of the material in Chapter 9, if the instructor is so inclined;
10. Chapter 11, although I rarely say much about root-locus in the course I teach;
11. Chapter 12, omitting Section 12.3 if robustness has not been covered in the earlier material;
12. perhaps some of the advanced PID synthesis methods of Chapter 13.

When I teach the introductory course, it is offered with a companion lab class. The lab course follows the lecture course in content, although it is somewhat more “down to earth.” Labs start out with the objective of getting students familiar with the ideas introduced in lectures, and by the end of the course, students are putting into practice these ideas to design controllers.

A more advanced course, having as prerequisite the material from the basic course, could be structured as follows:
1. thorough treatment of material in Chapter 2;
2. ditto for Chapter 3;
3. Bode’s Gain/Phase Theorem (Section 4.4.2) and uncertainty models (Section 4.5);
4. thorough treatment of signal and system norms from Section 5.3, proofs of Routh/Hurwitz criteria if one is so inclined, and Liapunov methods for stability (Section 5.4);
5. static state feedback, static output feedback, and dynamic output feedback (Section 6.4);
6. robust stability (Section 7.3);
7. design limitations in Chapter 9;
8. robust performance (Section 9.3);
9. Chapter 10, maybe omitting Section 10.4 on strong stabilisation;
10. basic loop shaping using robustness criterion (Section 12.3);
11. perhaps the advanced synthesis methods of Chapter 13;
12. Chapter 14;
13. Chapter 15.

\*Of course, someone teaching a traditional course is unlikely to be using this book.

The content

In Chapter 1 we engage in a loose discourse on ideas of a control theoretic nature. The value of feedback is introduced via a simple DC servo motor example using proportional feedback. Modelling and linearisation are also discussed in this chapter. From here, the book breaks up into three parts (plus appendices), with the presentation taking a rather less loose form.

Part I. System representations and their properties

Linear systems are typically represented in one of three ways: in the time domain using state space methods (Chapter 2); in the Laplace transform domain using transfer functions (Chapter 3); and in the frequency domain using the frequency response (Chapter 4). These representations are all related in one way or another, and there exist vocal proponents of one or the other representation. I do not get involved in any discussion over which representation is “best,” but treat each with equal importance (as near as I can), pointing out the innate similarities shared by the three models.

As is clear from the book’s subtitle, the treatment is single-input, single-output (SISO), with a very few exceptions, all of them occurring near the beginning of Chapter 2. The focus on SISO systems allows students to have in mind simple models. MIMO generalisations of the results in the book typically fall into one of two categories, trivial and very difficult. The former will cause no difficulty, and the latter serve to make the treatment more difficult than is feasible in an introductory text. References are given to advanced material.

Specialised topics in this part of the book include a detailed description of zero dynamics in both the state space and the transfer function representations. This material, along with the discussion of the properties of the transfer function in Section 3.3, have a rather technical nature. However, the essential ideas can be easily grasped independent of a comprehension of the proofs. Another specialised topic is a full account of Bode’s Gain/Phase Theorem in Section 4.4.2. This is an interesting theorem; however, time does not normally permit me to cover it in an introductory course.

A good understanding of the material in this first part of the book makes the remainder of the book somewhat more easily digestible. It has been my experience that students find this first material the most difficult.

Part II. System analysis

Armed with a thorough understanding of the three representations of a linear system, the student is next guided through methods for analysing such systems. The first concern in such a discussion should be, and here is, stability. A control design cannot be considered in any way successful unless it has certain stability properties. Stability for control systems has an ingredient that separates it from stability for simple dynamical systems. In control, one is often presented with a system that is nominally unstable, and it is desired to stabilise it using feedback. Thus feedback is another central factor in our discussion of control systems analysis. We are rather more systematic about this than is the norm. The discussion of signal flow graphs in Sections 6.1 and 6.2 is quite detailed, and some of this detail can be skimped. However, the special notion of stability for interconnected systems, here called IBIBO stability, is important, and the notation associated with it appears throughout the remainder of the book. The Nyquist plot criterion for IBIBO stability is an important part of classical control. Indeed, in Section 7.3 the ideas of the Nyquist plot motivate our discussion of robust stability. A final topic in control systems analysis is
performance, and this is covered in two chapters, 8 and 9, the latter being concerned with limitations on performance that arise due to features of the plant.

The latter of the two chapters on performance contains some specialised material concerning limitations on controller design that are covered in the excellent text of Seron, Braslavsky, and Goodwin [1997]. Also in this chapter is presented the “robust performance problem,” whose solution comprises Chapter 15. Thus Chapter 9 should certainly be thought of as one of special topics, not likely to be covered in detail in a first course.

**Part III. Controller design**  
The final part of the text proper is a collection of control design schemes. We have tried to present this material in as systematic a manner as possible. This gives some emphasis to the fact that in modern linear control, there are well-developed design methods based on a solid mathematical foundation. That said, an attempt has been made to point out that there will always be an element of “artistry” to a good control design. While an out of the box controller using some of the methods we present may be a good starting point, a good control designer can always improve on such a design using their experience as a guide. This sort of material is difficult to teach, of course. However, an attempt has been made to give sufficient attention to this matter.

This part of the book starts off with a discussion of the stabilisation problem.

**Part IV. Background and addenda**  
There are appendices reviewing relevant material in linear algebra, the matrix exponential, complex variables, and transforms. It is expected that students will have seen all of the material in these appendices, but they can look here to refamiliarise themselves with some basic concepts.

**What is not in the book**  
The major omission of the book is discrete time ideas. These are quite important in our digital age. However, students familiar with the continuous time ideas presented here will have no difficulty understanding their discrete time analogues. That said, it should be understood that an important feature in control is missing with the omission of digital control, and that instructors may wish to insert material of this nature.

This book is in its third go around. The version this year is significantly expanded from previous years, so there are apt to be many errors. If you find an error, no matter how small, let me know!

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