Please read the following points carefully before proceeding!

1. Please note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

2. You are allowed one 8′ × 11′ sheet of paper with handwritten notes on each side.

3. No calculators are allowed.

4. This exam paper is for both Math 439 and Math 836. Math 439 students make sure you do not do problems intended only for Math 836 students. These problems are clearly marked!

5. Please do not write in red.

6. Print your student number at the top of this page. Do not provide your name.

7. You have 3 hours to complete the exam.

8. Please read all questions carefully before answering them.

9. Explain your answers fully. The rule to use is that if your answer could have been guessed correctly, then it needs at least some justification. It is possible that zero marks will be awarded for a correct answer which has with it no evidence of how it was obtained.

10. Answer all questions on the exam paper—any work not on the exam paper will not be graded. I have tried to ensure that you have enough space on the exam paper to answer each question, but do not feel as if your answer needs to fill out the available space in order to be correct!

11. Marks per question are shown in square brackets after the problem number. The total is 100 for students enrolled in Math 439 and 115 students enrolled in Math 836.

12. Check that your question paper has 17 pages.

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For Math 439

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For Math 836
**P1.** *Only for students in Math 836*: Let $Q$ be the $n$-dimensional configuration manifold for an interconnected mechanical system and suppose that the system is subject to a linear velocity constraint defined by $n - 1$ linearly independent constraint equations.

(a) [7½ points] Show that the distribution $\mathcal{D}$ satisfying the constraint equations is integrable.

(b) [7½ points] Describe the orbit of the constraint distribution $\mathcal{D}$ through a point $q_0 \in Q$. 
P2. Suppose that, given two vector fields $X$ and $Y$ on a manifold $M$, one wishes to define a new vector field $[X,Y]$ by defining it in coordinates as

$$[X,Y] = \left( \frac{\partial Y^j}{\partial x^k} X^k - \frac{\partial X^j}{\partial x^k} Y^k \right) \frac{\partial}{\partial x^j}.$$ 

Do the following.

(a) [7½ points] By showing that the components obey the appropriate transformation rule, show that the definition of $[X,Y]$ makes sense.

Note: Since I am giving you the answer, the only points here are style points. So show me some style.

(b) [7½ points] Show that, given a function $f \in C^\infty(M)$,

$$\mathcal{L}_{[X,Y]} f = \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f.$$
**P3.** The following is a collection of short answer questions. Some tips:

1. I am looking for you to show me that you understand the essential feature of the question.
2. A correct answer might be very short.
3. You will be penalised for things you write that are incorrect or nonsensical. So the right answer embedded in a bunch of wrong stuff might get you zero marks.

Here are the questions.

(a) [7 points] Why does force take values in the cotangent bundle?
(b) [7 points] Why is acceleration, i.e., the second derivative of position with respect to time, not a tangent vector? How can one deal with acceleration in a precise way?
(c) [7 points] Why did we spend so much time talking about differential geometry in a course on mechanics?

*Note:* Answers containing excessive sarcasm or references to the instructor’s deranged mental state will not receive a lot of partial marks.
(d) [Bonus: 5 points] How would you respond to someone using philosophical arguments to justify a physical theory?
Consider a disk rolling in a vertical plane along terrain described by an infinitely differentiable function $f$. That is to say, if $(x, y)$ are Cartesian coordinates for the plane, then the disk rolls without slipping along the graph of the function $f$:

$$\text{graph}(f) = \{(x, f(x)) \mid x \in \mathbb{R}\}.$$  

See Figure 1. Here is some information.

\begin{figure}[h]
    \centering
    \includegraphics[width=0.5\textwidth]{figure1.png}
    \caption{A disk rolling through hills and valleys to the sound of birds chirping and fluffy white lambs bleating}
\end{figure}

1. You may assume that the radius of the disk and the function $f$ describing the terrain are such that the disk is always in contact with the ground at a single point.
2. It may be helpful to recall that the arclength along the graph of $f$ between $x_1$ and $x_2$ is given by

$$\int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2} \, dx.$$  

3. Although the disk rolls without slipping, this is \textit{not} a problem about linear velocity constraints.

Other assumptions you need to make and the physical constants needed, you should introduce as necessary, making sure you tell me as you go along. If you are unsure about any physical assumptions, make \textit{reasonable} assumptions yourself and proceed with these. Just tell me what you are doing.

(a) \textbf{[2 points]} On the figure above, draw natural spatial and body reference frames.
(b) \textbf{[2 points]} What is the free configuration manifold, $Q_{\text{free}}$, for the system?
(c) \textbf{[6 points]} Describe explicitly the submanifold $Q$ of $Q_{\text{free}}$ that corresponds to the admissible configurations of the system.
(d) \textbf{[4 points]} Provide a coordinate chart for $Q$, and, on the figure above, indicate the physical meaning of your coordinates.
(e) \textbf{[5 points]} Give the coordinate representation for the forward kinematic map $\Pi_1 : Q \to \text{SO}(3) \times \mathbb{R}^3$.
(f) \textbf{[4 points]} What is the inertia tensor for the disk about its centre of mass?
(g) \textbf{[7 points]} Use $\Pi_1$ to determine an expression for the kinetic energy metric of the system.
(h) [5 points] Write the equations of motion for the system in the absence of any applied forces.

(i) [5 points] The system is subject to a gravitational force as indicated by the vector $g$ in Figure 1. Write down the expression for the potential function in coordinates.

(j) [5 points] What is the force due to the potential function you derived in the preceding part of the question?

(k) [4 points] Add this force to the equations of motion you determined in part (h).
P5. [15 points] Suppose that a homogeneous spherical ball of radius $\rho$ rolls without slipping on a plane $P$. The configuration manifold is $Q = \text{SO}(3) \times \mathbb{R}^2$ and we denote a point in $Q$ by $(R, (x, y))$, where $(x, y)$ are Cartesian coordinates for the plane $P$. Thus a point in $TQ$ can be denoted by $((R, (x, y)), (\dot{R}, (\dot{x}, \dot{y})))$.

Determine the equations defining the constraint that the ball rolls without slipping. Express your equations in terms of $((R, (x, y)), (\dot{R}, (\dot{x}, \dot{y})))$. 