Please read the following points carefully before proceeding!

1. Please note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.
2. You are allowed one 8" × 11" sheet of paper with handwritten notes on each side.
3. No calculators are allowed.
4. This exam paper is for both Math 439 and Math 836. Math 439 students make sure you do not do problems intended only for Math 836 students. These problems are clearly marked!
5. Please do not write in red.
6. Print your student number at the top of this page. Do not provide your name.
7. You have 3 hours to complete the exam.
8. Please read all questions carefully before answering them.
9. Explain your answers fully. The rule to use is that if your answer could have been guessed correctly, then it needs at least some justification. **It is possible that zero marks will be awarded for a correct answer which has with it no evidence of how it was obtained.**
10. Answer all questions on the exam paper—any work not on the exam paper will not be graded. I have tried to ensure that you have enough space on the exam paper to answer each question, but do not feel as if your answer needs to fill out the available space in order to be correct!
11. Marks per question are shown in square brackets after the problem number. The total is 100 for students enrolled in Math 439 and 115 students enrolled in Math 836.
12. Check that your question paper has 19 pages.

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For Math 439

For Math 836
P1. Only for students in Math 836: Let $M = \mathbb{R} \times \mathbb{R}_{>0}$ (here $\mathbb{R}_{>0} = \{ x \in \mathbb{R} \mid x > 0 \}$) with $(x, y)$ Cartesian coordinates. On $M$ consider the Riemannian metric
\[ G = \frac{1}{y^2} (dx \otimes dx + dy \otimes dy). \]

Answer the following questions.

(a) [5 points] Compute the Christoffel symbols for the Levi-Civita connection associated with $G$.

(b) [5 points] Write down in coordinates the differential equations for the geodesics of the Levi-Civita connection.

(c) [5 points] Show that if $\gamma : I \to M$ is a geodesic (here $I \subset \mathbb{R}$ is an interval), then $\text{image}(\gamma)$ lies in a semicircle of the form
\[ \{ (x, y) \in \mathbb{R} \times \mathbb{R}_{>0} \mid (x - x_0)^2 + y^2 = R^2 \text{ for some } x_0 \in \mathbb{R}, \ R \in \mathbb{R}_{>0} \}. \]
P2. Let $M = \mathbb{R}^3$, let $(x, y, z)$ be Cartesian coordinates for $\mathbb{R}^3$, and consider the vector fields

$$X = \frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}. $$

Answer the following questions.

(a) [5 points] Compute $[X, Y]$.

(b) [5 points] For $t > 0$ compute $\Phi_{\sqrt{t}}^{-Y} \circ \Phi_{\sqrt{t}}^{-X} \circ \Phi_{\sqrt{t}}^{Y} \circ \Phi_{\sqrt{t}}^{X}(0, 0, 0)$.

(c) [5 points] Verify that

$$[X, Y](0, 0, 0) = \frac{d}{dt} \bigg|_{t=0} \Phi_{\sqrt{t}}^{-Y} \circ \Phi_{\sqrt{t}}^{-X} \circ \Phi_{\sqrt{t}}^{Y} \circ \Phi_{\sqrt{t}}^{X}(0, 0, 0).$$
P3. Let $M = \mathbb{R}^2$, let $(x, y)$ be Cartesian coordinates for $\mathbb{R}^2$, and consider the affine connection $\nabla$ on $\mathbb{R}^2$ with nonzero Christoffel symbols

$$\Gamma_{xx}^x = -4y, \quad \Gamma_{yy}^x = x^2, \quad \Gamma_{xx}^y = -2.$$ 

Answer the following questions.

(a) [6 points] In coordinates, write the differential equations for the geodesics of $\nabla$.

(b) [6 points] Determine which of the following curves $\gamma_1$ and $\gamma_2$, represented in coordinates, is a geodesic of $\nabla$:

$$\gamma_1: t \mapsto (t, t^2), \quad \gamma_2: t \mapsto (2t^2, -3t).$$
P4. In this problem, we consider a disk rolling on the inner surface of a circular track, as shown in Figure 1. We ask that the disk have radius \( \rho \), that the circular track have radius \( \rho/2 \), and that the disk have an axis of symmetry about the axis orthogonal to the plane of motion. Other physical constants, you should introduce as necessary, making sure you tell me what they are.

If you are unsure about any physical assumptions, make reasonable assumptions yourself and proceed with these. Just tell me what you are doing.

(a) [2 points] On the figure above, draw natural spatial and body reference frames.

(b) [2 points] What is the free configuration manifold, \( Q_{\text{free}} \), for the system?

(c) [5 points] Describe explicitly the submanifold \( Q \) of \( Q_{\text{free}} \) that corresponds to the admissible configurations of the system.

(d) [4 points] Provide a coordinate chart for \( Q \), and, on the figure above, indicate the physical meaning of your coordinates.

(e) [5 points] Give the coordinate representation for the forward kinematic map \( \Pi_1 : Q \to \text{SO}(3) \times \mathbb{R}^3 \).

(f) [4 points] What is the inertia tensor for the disk about its centre of mass?

(g) [6 points] Use \( \Pi_1 \) to determine an expression for the kinetic energy metric for the system.

(h) [4 points] Write the equations of motion for the system in the absence of any applied forces.

(i) [4 points] The system is subject to gravity in the direction indicated in Figure 1. What is the representation of the gravitational force in Newtonian mechanics (i.e., as a vector in \( \mathbb{R}^3 \))?

(j) [4 points] Using the forward kinematic map, determine the Lagrangian representation of this force.

(k) [3 points] Add this force to the equations of motion you determined in part (h).
P5. Let $Q = \mathbb{R}^3$ and on $Q$ consider an affine connection $\nabla$ whose nonzero Christoffel symbols are given by

$$\Gamma^z_{xy} = 1,$$

where $(x, y, z)$ denote Cartesian coordinates for $\mathbb{R}^3$. Also suppose that the system has input vector fields

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y},$$

thus defining an affine connection control system $\Sigma = (Q, \nabla, \mathcal{V}, \mathbb{R}^m)$.

(a) [3 points] In the coordinates $(x, y, z)$ write the differential equations which govern trajectories for the system.

(b) [7 points] Show that the system is not reducible to a driftless system.

(c) [7 points] Find two decoupling vector fields for $\Sigma$.

(d) [7 points] Is $\Sigma$ kinematically controllable?

(e) [3 points] Can you deduce (i.e., make an educated guess) from the governing differential equations whether the system is controllable from $(0, 0, 0)$ with zero initial velocity?

(f) [3 points] If we instead had defined $\nabla$ as having nonzero Christoffel symbols

$$\Gamma^z_{xx} = 1,$$

deduce from the governing differential equations whether $\Sigma$ is controllable or not.
HAVE A GREAT SUMMER!