Chapter 2

Lagrangian mechanics

After understanding some essential concepts in Newtonian mechanics we are in a position to get to Lagrangian mechanics proper. Lagrangian mechanics can be quite abstract, but to be really precise, some of this abstraction is necessary. The starting point for Lagrangian mechanics in the modern framework [Abraham and Marsden 1978, Arnol’d 1989] is with a configuration space that is a “differentiable manifold.” From here one is able to construct the objects of Lagrangian mechanics as defined on a differentiable manifold. While I feel that such a treatment has much to recommend it, it does necessitate something of a buildup, and at this level this buildup may not be appropriate. Thus our approach is to (1) identify the critical concept of coordinate invariance. The idea with coordinate invariance is that, at least when dealing with general considerations (as opposed to a specific example), one should never deal with things that are not coordinate invariant. In order to see what that might mean, we begin at the beginning.

2.1 Configuration spaces Suppose that we are given a collection of particles and rigid bodies, perhaps interconnected in some way. We wish to assign to this mechanical entity a set that describes its configuration. That is to say, we wish to find a set whose points are in one-to-one correspondence with the set of configurations of the system. There is generally speaking something of an art to doing this, and we shall demonstrate how this might be done via examples. As we introduce the examples, we will introduce the notation one often encounters when trying to assign a set to the collection of configurations for a system.

2.1.1 Examples 1. Consider a mass moving in a plane and constrained to move on a circle of radius $r$ in that plane. We need to parse this description with the language we know. First of all, when we say something is moving in a plane, what we really mean is that it is constrained to move in a three-dimensional Galilean sub-spacetime $\mathcal{E}$ of a big Galilean spacetime $\mathcal{F} = (\mathcal{E}, V, q, \tau)$. Let $U$ be the subspace of $V$ upon which $\mathcal{F}$ is modelled. We then choose a coordinate system $\phi$ for $\mathcal{E}$ that maps $\mathcal{F}$ to the canonical 3-dimensional Galilean sub-spacetime $\mathcal{F}_3$. If the mass is further constrained to move on a circle, this means that for each collection $\mathcal{E}(s)$ of simultaneous events there exists a point $x_s \in \mathcal{E}(s)$ with the property that the particle must lie on the circle

$$\{ x_s + ru | u \in U, \; g(u, u) = 1 \}.$$ 

It is not clearly stated, but we may suppose that with respect to some observer, the centre of the circle is stationary. This means that we may select the coordinate system $\phi$ so that all points $x_s$ get mapped to $0 \in \mathbb{R}^3$. Therefore, we have reduced ourselves to the situation that we would have guessed by the initial description:

The particle moves on the circle of radius $r$ centred at $(0, 0) \in \mathbb{R}^2$.

We still need to describe the configuration space. Let us define

$$S^n = \{ x \in \mathbb{R}^{n+1} | \|x\| = 1 \}$$

to be the $n$-dimensional sphere. We claim that we may take $S^1$ as the configuration space for the system. Indeed, the particle moves on the circle of radius $r$, and there is certainly a simple correspondence between points on the circle of radius $r$ and those on the circle of radius 1 (see Figure 2.1). Thus we select $Q = S^1$ as the configuration space for the system.

Why not choose the configuration space to be the sphere of radius $r$ rather than the sphere of radius 1? There is no reason not to do this. However, the configuration space is typically chosen to be some dimensionless abstract object, and the physics of the problem, in this case the exact radius of the circle, are shoved into other aspects of the problem description, as we shall see.

2. Our next mechanical system is depicted in Figure 2.2. We have two rigid bodies, each
moving in the same plane, one with a stationary base point, and connected to the base of the other link at its tip. Let us see how to define a configuration space for this system. The links move in the same plane so there exists a three-dimensional Galilean sub-spacetime $\mathcal{F}$ of $\mathcal{E}$ to which the dynamics restricts. We take a coordinate system $\phi$ that maps $\mathcal{F}$ to $\mathcal{F}_1$. The links then move around in a nice copy of $\mathbb{R}^2$. As one of the links has a fixed point, let us choose our coordinate system so that this point is at the origin in $\mathbb{R}^2$. Now, to fix the position of the link whose base point is fixed, we need only specify its angle relative to, say, the $x$-axis in $\mathbb{R}^2$. Note that one we know the position of the link with the fixed base point, we also know the position of one point on the other link since we are told that they interact with our machinations with Galilean spacetimes in Chapter 1. Note that in none of the above examples have we said anything about the system other than its configuration space. We have said nothing about the quantitative geometry of the system (e.g., the radius $r$ of the circle in Example 2.1–1 or the lengths of the links in Example 2.1–2). We have also said nothing about the inertial properties of the system or the external forces acting on the system. As a general rule, a good idea with a mechanical problem is to first determine its configuration space. It is with respect to the configuration space that everything else is done, as we shall see.

### 2.1.2 Coordinates

In the preceding examples we identified sets that describe the configurations of a system. Note that in none of the cases was the configuration space Euclidean. That is to say, it is not possible to establish a smooth correspondence with the configuration spaces and open subsets of $\mathbb{R}^n$ for some appropriate $n$. This is problematic. To describe the dynamics of these systems, we need to be able to differentiate things, and if we are only able to differentiate on Euclidean spaces, then it is not clear how to proceed to deal with the configuration spaces in the examples. But on the other hand, what’s the problem? Certainly in Example 2.1.1–1 you would just choose an angular coordinate and proceed. Similarly, in Example 2.1.1–2 you’d assign two angular coordinates and proceed. Here, however, you begin to see that you have some choices to make. Should you measure the angle of the second link with respect to the first link, or with respect to the “horizontal”? Moving
on to Example 2.1.1–3 things seem altogether less transparent. What coordinates should you use? The fact of the matter is that things were not so transparent in Example 2.1.1–1, but you got over it because you were familiar with the configuration space. In Example 2.1.1–2 you saw you had a choice to make, but any hesitation about how things might depend on this choice were washed away by your anxiousness to proceed. However, for Example 2.1.1–3 it is not really clear how to begin.

We wish to systematise the above discussion by clearly identifying the process you readily undertook in the easy examples. By understanding these, you at least know what you need to do to start with Example 2.1.1–3. The idea is that we make somewhat precise the idea of choosing coordinates. Our definition here actually destructs under close scrutiny, but it will suffice as long as we are gentle with it.

2.1.2 Definition Let Q be a configuration space for a mechanical system. A coordinate chart for Q is a pair (U, φ) where

(i) U is a subset of Q and

(ii) φ: U → Uφ is a map from U to an open subset Uφ of \( \mathbb{R}^n \), for some n, that is a bijection (i.e., is one-to-one and onto).

Let us see if we can make sense of this for the three systems of Example 2.1.1.

2.1.3 Examples 1. Example 2.1.1–1 cont’d: We wish to choose a coordinate chart (U, φ) for Q = \( S^1 \). We take

\[ U = \{ -1, 0 \} \subseteq \mathbb{R}^1 \]

and

\[ \phi(x, y) = \tan(x, y). \]

Here atan: \( \mathbb{R}^2 \setminus \{0, 0\} \rightarrow (-\pi, \pi] \) is the usual angle measured so that atan(x, 0) = 0 for x > 0. Note here that Uφ = (−π, π) is indeed an open subset of \( \mathbb{R}^1 \).

2. Example 2.1.1–2 cont’d: Here we choose a coordinate chart for Q = T^2. We take

\[ S_1 = \{ (x_1, y_1), (x_2, y_2) \in Q \mid x_1 = -1 \} \]

\[ S_2 = \{ (x_1, y_1), (y_2, y_2) \in Q \mid x_2 = -1 \} , \]

and then

\[ U = Q \setminus (S_1 \cup S_2), \quad \phi((x_1, y_1), (x_2, y_2)) = (\tan(x_1, y_1), \tan(x_2, y_2)). \]

Note here that Uφ = (−π, π) × (−π, π).

3. Example 2.1.1–3 cont’d As expected, choosing a coordinate chart for SO(3) is not so easy to do. What’s more, it is regarded in some circles as a silly thing to do as often there are better ways to handle systems involving SO(3) than choosing coordinates. Nonetheless, we shall illustrate how one can do that, just so that we might illustrate that it is possible. We shall provide coordinates for a neighbourhood of I_3 in SO(3) by using Euler angles. Other coordinate charts are possible, and we refer to [Murray, Li, and Sastry 1994] for details.

Let us first be formal, and then we shall give the intuition behind what we do. Let R be a matrix in SO(3) which is “close” to, but not equal to, I_3. We shall be clear about how close shortly. Let us write the components for R as R_{ij}, i, j = 1, 2, 3 where first index is the row index, and the second the column index. We then define

\[ \beta \in (0, \pi), \quad \alpha \in (-\pi, \pi), \quad \gamma \in (-\pi, \pi) \]

by

\[ \beta = \tan(R_{33}, \sqrt{R_{11}^2 + R_{22}^2}), \quad \alpha = \tan(R_{13}/\sin\beta, R_{23}/\sin\beta), \]

\[ \gamma = \tan(-R_{31}/\sin\beta, R_{32}/\sin\beta). \]

This then defines a map φ from some subset U of SO(3) to \( \mathbb{R}^3 \), thereby defining a coordinate chart. The subset of SO(3) is given by inverting the relations (2.2) to give the matrix entries, and doing so gives the matrix

\[ \begin{bmatrix}
\cos\alpha \cos\beta \cos\gamma & -\cos\alpha \cos\beta \sin\gamma & -\sin\alpha \cos\gamma \\
\cos\alpha \sin\beta \cos\gamma & \cos\alpha \sin\beta \sin\gamma & \cos\alpha \sin\gamma \\
-\sin\beta \cos\gamma & -\sin\beta \sin\gamma & \cos\beta
\end{bmatrix} . \]

Thus U is the subset of all matrices in SO(3) of the form (2.3) where the numbers α, β, and γ satisfy the relations (2.1). The map φ: U → \( \mathbb{R}^3 \) is given by

\[ \phi(R) = (\tan(R_{33}, \sqrt{R_{11}^2 + R_{22}^2}), \tan(R_{13}/\sin\beta, R_{23}/\sin\beta), \tan(-R_{31}/\sin\beta, R_{32}/\sin\beta)). \]

Note that this map does not include I_3 in the chart domain U, and so that is kind of lucky. The fact of the matter is, we will never use coordinates for SO(3). It is far more convenient when dealing with SO(3) to proceed as we did in Chapter 1 with rigid body dynamics.}

If Q is a configuration space with (U, φ) a coordinate chart, we will often pretend that Uφ = φ(U) ⊂ \( \mathbb{R}^n \) is the configuration manifold. But note that this is just pretend. But when we do this, we will often write a point q ∈ Q by its image under φ that we might denote φ(q) = (q^1, ..., q^n). Thus (q^1, ..., q^n) are coordinates in the coordinate chart (U, φ). We will think of functions on Q as being functions of the real variables (q^1, ..., q^n). In this way we reduce ourselves to something with which we are familiar. Note that we put superscripts on our coordinates! That is, q^2 is the second coordinate, not the square of q. We do this because tradition dictates that this is what we should do. Other objects we encounter will have subscripts as labels, and one can ascertain the character of something by looking at where its indices are located. But we will get to this in time.

The idea of Definition 2.1.2 is that one establishes an exact correspondence of a subset of one’s configuration space with something one can deal with, an open subset of Euclidean space. In doing so, one makes it so that one can deal with these portions of the configuration space in a way with which one is familiar. Everything seems okay, but the problem is that it is generally not possible to find a chart (U, φ) where U = Q. Thus to cover the entire configuration space we may need two or more charts. Generally, these charts may overlap. Thus we may have a situation where a subset of the configuration space may have two or more coordinate charts describing it. These coordinate charts should have some properties relative to one another.
2.1.4 Definition Let $Q$ be a configuration space for a mechanical system and let $(U_1, \phi_1)$ and $(U_2, \phi_2)$ be coordinate charts for $Q$. The coordinate charts are compatible if the map

$$\phi_{12}: \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

$$x \mapsto \phi_2 \circ \phi_1^{-1}(x)$$

satisfies

(i) $\phi_{12}$ is a bijection,

(ii) $\phi_{12}$ is differentiable, and

(iii) $\phi_{12}^{-1}$ (that exists by (i)) is differentiable.

If there exists compatible coordinate charts $(U_1, \phi_1), \ldots, (U_k, \phi_k)$ so that $U_1 \cup \cdots \cup U_k = Q$ and so that all maps $\phi_1, \ldots, \phi_k$ take their values in $\mathbb{R}^n$, then the dimension of $Q$ is $n$. A mechanical system with an $n$-dimensional configuration space has $n$ degrees of freedom. \(\square\)

In the following, we shall consider configuration spaces with a well-defined dimension, and we will reserve the letter $n$ to represent the dimension of an arbitrary configuration space. Thus, below, whenever you see an $n$, it will be the dimension of the configuration manifold that is lying about at the time.

A picture to have in mind with respect to this definition is provided in Figure 2.3.

![Figure 2.3 Compatability of coordinate charts](image)

Our first coordinate chart is the “obvious” one. We define a chart $(U_1, \phi_1)$ by $U_1 = Q$ and $\phi(x, y) = (x, y)$. Note that we are able to choose a chart that covers all of $Q$. This is typically not the case.

Now we define $(U_2, \phi_2)$ by

$$U_2 = \mathbb{R}^2 \setminus \{(x, y) \mid x \leq 0\}, \quad \phi_2(x, y) = (\sqrt{x^2 + y^2}, \text{atan}(x, y)).$$

The coordinate chart is illustrated in Figure 2.4. These coordinates are, of course, the usual polar coordinates you are all familiar with. We are simply being more careful with the description.

Let us now see whether these two coordinate charts are compatible in the sense of Definition 2.1.4. We have

$$\phi_1(U_1 \cap U_2) = \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\}$$

$$\phi_2(U_1 \cap U_2) = \{(r, \theta) \mid r > 0, \theta \in (-\pi, \pi)\}.$$ 

Note that we use coordinates $(x, y)$ for $\phi_1(U_1 \cap U_2)$ and coordinates $(r, \theta)$ for $\phi_2(U_1 \cap U_2)$. The map $\phi_{12}$ is then computed to be

$$\phi_{12}(x, y) = (\sqrt{x^2 + y^2}, \text{atan}(x, y)).$$

This map is certainly one-to-one and onto—it establishes a correspondence between points in $\phi_1(U_1 \cap U_2)$ and points in $\phi_2(U_1 \cap U_2)$. Thus $\phi_{12}$ satisfies condition (i) of Definition 2.1.4. Keeping in mind that the domain of $\phi_{12}$ is $\phi_1(U_1 \cap U_2)$, we also see that $\phi_{12}$ satisfies condition (ii) of Definition 2.1.4 since its Jacobian is

$$D\phi_{12}(x, y) = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} \end{bmatrix}.$$ 

This matrix has continuous entries, so $\phi_{12}$ is differentiable. The inverse of $\phi_{12}$ is readily determined to be

$$\phi_{12}^{-1}(r, \theta) = (r \cos \theta, r \sin \theta).$$
so that
\[ D\phi^{-1}_{12}(r, \theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix}. \]

Since this matrix has continuous entries, \( \phi^{-1}_{12} \) is differentiable, and so \( \phi_{12} \) satisfies condition (iii) of Definition 2.1.4.

All of the above shows that the two charts \((U_1, \phi_1)\) and \((U_2, \phi_2)\) are compatible. \( \square \)

With Definition 2.1.4 we are converging to the proper object to describe the configuration space of a mechanical system: a differentiable manifold. However, having gotten this close, we are going to back off and generally work in a fixed coordinate chart for \( Q \), and ignore the fact that it may not cover all of \( Q \). At various times, however, we will point out the effects of demanding coordinate charts that are compatible.

### 2.1.3 Functions and curves

One can talk of all manner of objects defined on a configuration space once one has the idea of coordinate charts. We shall start by talking about some of the simpler notions, those of functions and curves.

Let \( Q \) be a configuration manifold. A function \( f: Q \to \mathbb{R} \) is **differentiable** if for each \( q \in Q \) and for each coordinate chart \((U, \phi)\) with \( q \in U \), the function \( f_\phi: U \to \mathbb{R} \) defined by

\[ f_\phi(q) = f(\phi^{-1}(q)) \]

is differentiable. The idea is very simple, of course. To test differentiability of a function on \( Q \), one simply tests it in a coordinate chart. One can easily check that one does not have to check the condition for every chart, but that it suffices to find one for each point. Also, although our definition is one for differentiability, one can check for any order of differentiability, including simple continuity by our definition.

Now let us talk about curves. A **curve** is a map \( c: I \to Q \) from an interval \( I \subset \mathbb{R} \) with the property that for each \( t_0 \in I \) and for every coordinate chart \((U, \phi)\) with \( c(t_0) \in U \), the map \( \phi_\phi: I \to U \) defined by \( \phi_\phi(t) = c(\phi(t)) \) is differentiable. Here \( I \subset I \) is an interval with the property that \( t_0 \in I \) and \( t \in U \) for every \( t \in I \).

When representing both functions and curves in a single given coordinate chart, we will often engage in an abuse of notation and write a function as \( f(q^1, \ldots, q^n) \) and a curve as \( t \mapsto (q^1(t), \ldots, q^n(t)) \). The abuse of notation is that we are pretending that the single coordinate chart suffices to define the function and the curve, and this may not be the case.

### 2.2 Vector fields, one-forms, and Riemannian metrics

Lagrangian mechanics involves certain objects that transform in certain ways when one changes from a coordinate chart to another compatible one. There is a theory behind how one talks about such objects, and in this section we give a very cursory introduction to such things. For a more thorough account, we refer to [Abraham, Marsden, and Ratiu 1988].

Although there is some significant formalism to what we do here, it is entirely probable that the material we cover in this section is familiar, and we shall try as much as possible to make contact with more commonplace notions.

#### 2.2.1 Tangent vectors, tangent spaces, and the tangent bundle

After talking about configuration spaces, the next thing we must do is talk sensibly about “velocities” on configuration spaces. Again, things are complicated here by the fact that a configuration space is not a nice Euclidean space, and so our presentation of velocities relies on the use of coordinate charts.

Let \( Q \) be a configuration space with \((U_1, \phi_1)\) a coordinate chart for \( Q \) and \( q \in U_1 \). A **\( \phi_1 \)-tangent vector at \( q \)** is an element \( X_{\phi_1} \in \mathbb{R}^n \). This is too boring. What happens if we have another coordinate chart \((U_2, \phi_2)\) where \( q \in U_2 \)? Well, we can certainly talk about a \( \phi_2 \)-tangent vector \( X_{\phi_2} \) at \( q \). What we want is a way to say that a \( \phi_1 \)-tangent vector and a \( \phi_2 \)-tangent vector are represent “the same thing.” To motivate how to do this, let us consider a curve \( c: [-\epsilon, \epsilon] \to Q \) with the property that \( c(0) = q \). Suppose that the curve in the chart \((U_1, \phi_1)\) is represented by \( t \mapsto (q^1(t), \ldots, q^n(t)) \) and that the curve in the chart \((U_2, \phi_2)\) is represented by \( t \mapsto (\tilde{q}^1(t), \ldots, \tilde{q}^n(t)) \). We wish to talk about the “velocity” of the curve \( c \) as it passes through \( q = c(0) \). In the chart \((U_1, \phi_1)\) this velocity is represented by

\[ (q^1(\dot{t}), \ldots, q^n(\dot{t})) \in \mathbb{R}^n, \]

and in the chart \((U_2, \phi_2)\) it is represented by

\[ (\tilde{q}^1(\dot{t}), \ldots, \tilde{q}^n(\dot{t})) \in \mathbb{R}^n. \]

However, the chain rule dictates that

\[ \dot{\tilde{q}}(0) = \sum_{i=1}^n \frac{\partial \tilde{q}^i}{\partial q^j}(0) \dot{q}^j(0), \quad i = 1, \ldots, n. \]  

(2.4)

Before proceeding with our discussion proper, let us introduce an important notational convention. We shall write

\[ \sum_{j=1}^n \frac{\partial \tilde{q}^i}{\partial q^j}(0) \dot{q}^j(0) = \frac{\partial \tilde{q}^i}{\partial q^j}(0) \dot{q}^j(0). \]

thus omitting the summation sign. The idea is that whenever you see a repeated index, one of which is a superscript and the other of which is a subscript, then summation will be implied, unless otherwise stated. Note that the \( j \) in \( \frac{\partial \tilde{q}^i}{\partial q^j} \) is a superscript in the denominator; these are regarded as subscripts! The details regarding the rules of the summation convention are included in Appendix A. Let us now proceed. From (2.4) we see that if we want a \( \phi_1 \)-tangent vector \( X_{\phi_1} \) and a \( \phi_2 \)-tangent vector \( X_{\phi_2} \) to be “the same,” we should require that

\[ X^i_{\phi_2} = \frac{\partial \tilde{q}^i}{\partial q^j}(0) X^j_{\phi_1}. \quad i = 1, \ldots, n. \]

We note that the matrix with components \( \frac{\partial \tilde{q}^i}{\partial q^j} \), \( i, j = 1, \ldots, n \), is none other than the Jacobian matrix \( D\phi_{12}(\phi_1(q)) \). Thus we say that \( X_{\phi_1} \) and \( X_{\phi_2} \) are **equivalent** if \( X_{\phi_2} = D\phi_{12}(\phi_1(q)) \cdot X_{\phi_1} \).

We may now formally define what we mean by a tangent vector.

#### 2.2.2 Definition

Let \( Q \) be a configuration space with \( q \in U \). A **tangent vector at \( q \)** is a pair \((X, S)\) where \( X \in \mathbb{R}^n \) is a \( \phi \)-tangent vector at \( q \) for some chart \((U, \phi)\) and

\[ S = \{(X, (U, \phi)) \mid X \text{ is a } \phi \text{-tangent vector at } q \text{ that is equivalent to } X \text{ for some chart } (U, \phi) \}. \]
The collection of all tangent vectors at \( q \) is denoted \( T_q Q \) and is called the **tangent space at** \( q \). The collection

\[
\bigcup_{q \in Q} T_q Q
\]

of all tangent spaces is called the **tangent bundle** and denoted \( TQ \).

Having stated this formal definition of a tangent vector, we will almost never use it. We will often refer to a tangent vector by writing it in a specific set of coordinates. Indeed, even its formal definition relies on an initial choice of coordinate chart for its definition. Some notation for writing tangent vectors is convenient. Let \((U, \phi)\) be a coordinate chart with coordinates \((q^1, \ldots, q^n)\). For each \( q \in U \) there are \( n \) distinguished tangent vectors given by

\[
X_i = e_i, \quad i = 1, \ldots, n,
\]

where \( e_i \) is the \( i \)th standard basis vector for \( \mathbb{R}^n \). We shall write \( X_i = \frac{\partial}{\partial q^i} \bigg|_q \). This may seem like strange notation, but as we shall see, it is extremely convenient. It is then possible to write any tangent vector \( X \) at \( q \) as

\[
X = X^i \frac{\partial}{\partial q^i}
\]

for some \( X^1, \ldots, X^n \in \mathbb{R} \) called the **components** of \( X \). Thus the tangent vectors \( \{ \frac{\partial}{\partial q^1} \bigg|_q, \ldots, \frac{\partial}{\partial q^n} \bigg|_q \} \) form a basis for the tangent space \( T_q Q \).

Let us see how our basis vector is changed by changing coordinate charts. In the following result we let \((q^1, \ldots, q^n)\) be coordinates in a chart \((U_1, \phi_1)\) and \((q^1, \ldots, q^n)\) be coordinates in a chart \((U_2, \phi_2)\).

**Lemma 2.2.2** Let \( \frac{\partial}{\partial q^i} \bigg|_q = \frac{\partial}{\partial q^i} \bigg|_q(\phi_1(q)) \frac{\partial}{\partial q^i} \bigg|_q \), \( i = 1, \ldots, n \).

**Proof** Let us work only in the intersection of the chart domains, and so take \( U = U_1 \cap U_2 \). At \( q \in U \) we have the tangent vector \( \frac{\partial}{\partial q^i} \bigg|_q \) for fixed \( i \in \{1, \ldots, n\} \). We also have the basis of tangent vectors \( \{ \frac{\partial}{\partial \phi_1} \bigg|_q, \ldots, \frac{\partial}{\partial \phi_n} \bigg|_q \} \). We can then write

\[
\frac{\partial}{\partial q^k} \bigg|_q = \xi^k \frac{\partial}{\partial q^i} \bigg|_q
\]

for a collection of numbers \( \xi^1, \ldots, \xi^n \) on \( U \). The tangent vectors on the left and right side of the equation are equivalent. This means that

\[
D\phi_2 \phi_1(q) \frac{\partial}{\partial q^i} \bigg|_q = \xi^i(q) \frac{\partial}{\partial q^i} \bigg|_q.
\]

Now note that the matrix representing the Jacobian \( D\phi_2 \phi_1(q) \) is exactly \( \frac{\partial}{\partial \phi} \), \( i, k = 1, \ldots, n \). To apply the Jacobian to \( \frac{\partial}{\partial q} \), we apply the matrix \( \frac{\partial}{\partial \phi} \) to the components of the vector field \( \frac{\partial}{\partial \phi} \). But the components of this vector field are

\[
\delta^k_i = \begin{cases} 
1, & i = k \\
0, & i \neq k.
\end{cases}
\]

Therefore

\[
\xi^i(q) = \frac{\partial q^i}{\partial \phi_k} \bigg|_q \delta^k_i = \frac{\partial q^i}{\partial \phi_k} \bigg|_q,
\]

which is as claimed.

Note that this result agrees with what the notation looks like! That is to say, if you were to treat the partial derivative notation in terms of differential operators rather than vector fields, it would look just like as stated in the lemma.

Let’s see how this plays out in a simple example.

**Example 2.2.3** (Example 2.1.5 cont’d) Recall that we had defined two coordinate charts on \( Q = \mathbb{R}^2; (U_1, \phi_1) \) were the standard Cartesian coordinates, and \((U_2, \phi_2)\) were polar coordinates. We shall define tangent vectors in each coordinate chart, and show that they are equivalent.

Let \( q = (x, y) \) so that \( \phi_1(q) = (x, y) \) and \( \phi_2(q) = (\sqrt{x^2 + y^2}, \text{atan}(x, y)) \). We define the \( \phi_1 \)-tangent vector at \( q \) by

\[
X_{\phi_1} = -y \frac{\partial}{\partial x} \bigg|_q + x \frac{\partial}{\partial y} \bigg|_q,
\]

and the \( \phi_2 \)-tangent vector by

\[
X_{\phi_2} = \frac{\partial}{\partial \theta} \bigg|_q.
\]

We shall show that these tangent vectors are equivalent. By Lemma 2.2.2 we have

\[
\frac{\partial}{\partial \theta} \bigg|_q = \frac{\partial}{\partial x} \bigg|_q \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \bigg|_q \frac{\partial y}{\partial \theta}
\]

\[
= \cos \theta \frac{\partial}{\partial x} \bigg|_q + \sin \theta \frac{\partial}{\partial y} \bigg|_q
\]

\[
= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} \bigg|_q + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y} \bigg|_q,
\]

and

\[
\frac{\partial}{\partial \theta} \bigg|_q = -r \sin \theta \frac{\partial}{\partial x} \bigg|_q + r \cos \theta \frac{\partial}{\partial y} \bigg|_q
\]

\[
= -y \frac{\partial}{\partial x} \bigg|_q + x \frac{\partial}{\partial y} \bigg|_q.
\]

Therefore, we directly see that

\[
X_{\phi_2} \frac{\partial}{\partial \theta} \bigg|_q = -y \frac{\partial}{\partial x} \bigg|_q + x \frac{\partial}{\partial y} \bigg|_q = X_{\phi_1},
\]

which means that the two tangent vectors are indeed equivalent. Understand, however, that \( X_{\phi_1} \) is defined on \( U_1 \) which is all of \( Q \) in this example, while \( X_{\phi_2} \) is only defined on the strict subset \( U_2 \) of \( Q \).

Note that in Cartesian coordinates the vectors are tangent to circles, so it makes sense that in polar coordinates, the tangent vectors would have only one component in the \( \theta \) direction. □

From Lemma 2.2.2 it also follows that we may determine from one another the components of the same vector field represented in different coordinates.
2.2.4 Lemma Let \( X \) be a tangent vector at \( q \in Q \) and let \((q^1, \ldots, q^n)\) and \((\tilde{q}^1, \ldots, \tilde{q}^n)\) be coordinates for \( Q \). If \( X^1, \ldots, X^n \) are the components for \( X \) in the coordinates \((q^1, \ldots, q^n)\), then
\[
\frac{\partial q^i}{\partial X^j} (\phi_i(q)) X^j, \ldots, \frac{\partial q^n}{\partial X^j} (\phi_i(q)) X^j
\]
are the components of \( X \) in the coordinates \((\tilde{q}^1, \ldots, \tilde{q}^n)\).

**Proof** Let \( \tilde{X}^1, \ldots, \tilde{X}^n \) be the components of \( X \) in the coordinates \((\tilde{q}^1, \ldots, \tilde{q}^n)\). We then have
\[
\tilde{X}^i \bigg|_q = X^j \frac{\partial \tilde{q}^i}{\partial q^j} \bigg|_q
\]
on the intersection of the two chart domains. However, from Lemma 2.2.2 this means that
\[
\tilde{X}^i \frac{\partial q^j}{\partial \tilde{q}^i} (\phi_i(q)) \frac{\partial q^j}{\partial \tilde{q}^i} \bigg|_q = X^j \frac{\partial q^j}{\partial q^j} \bigg|_q.
\]
Since the vector fields \( \{ \frac{\partial}{\partial q^1} |_q, \ldots, \frac{\partial}{\partial q^n} |_q \} \) are linearly independent, this means that \( \tilde{X}^j = \frac{\partial \tilde{q}^j}{\partial q^j} X^i \), as claimed.

Let us see how this works out in our polar coordinate example.

2.2.5 Example (Example 2.1.5 cont’d) Again we take \((U_1, \phi_1)\) to be Cartesian coordinates for \( Q = \mathbb{R}^2 \) and we take \((U_2, \phi_2)\) to be polar coordinates for \( Q \). Also again, we take the vector field \( X \) with the property that
\[
X_{q_1} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad X_{q_2} = \frac{\partial}{\partial \theta}.
\]
We take \((x,y)\) as the coordinates \((q^1, q^2)\) and \((r, \theta)\) as the coordinates \((\tilde{q}^1, \tilde{q}^2)\). Thus the components of \( X \) in the coordinates \((x,y)\) are \((-y, x)\) and the components in the coordinates \((r, \theta)\) are \((0, 1)\). Following Lemma 2.2.4 we have
\[
\frac{\partial r}{\partial x} (-y) + \frac{\partial r}{\partial y} (x) = \frac{x}{\sqrt{x^2 + y^2}} (-y) + \frac{y}{\sqrt{x^2 + y^2}} (x) = 0,
\]
and
\[
\frac{\partial \theta}{\partial x} (-y) + \frac{\partial \theta}{\partial y} (x) = -\frac{y}{x^2 + y^2} (-y) + \frac{x}{x^2 + y^2} (x) = 1.
\]
Things are as they should be. \(\square\)

We will very often wish to think of \( TQ \), the collection of all tangent vectors, as an object onto itself, with its own set of coordinates. To set about doing this, we let \((U, \phi)\) be a coordinate chart for \( Q \) with coordinates \((q^1, \ldots, q^n)\). Let us agree to write a typical tangent vector in \( T_qQ \) as \( v_q \), thereby emphasizing the fact that it is “attached” to the point \( q \). For any \( q \in Q \) and any \( v_q \in T_qQ \), we may write
\[
v_q = v^i \frac{\partial}{\partial q^i} \bigg|_q
\]
for some \( v^1, \ldots, v^n \in \mathbb{R} \). In this way, for any \( q \in Q \) and any \( v_q \in T_qQ \), we can assign a unique set of coordinates \((q^1, q^2, v^1, \ldots, v^n)\). We call these the natural coordinates for \( TQ \) associated with the coordinate chart \((U, \phi)\). Thus \( TQ \), the set of all configurations and all velocities, has dimension \( 2n \). If we have an overlapping set of coordinates \((\tilde{q}^1, \ldots, \tilde{q}^n)\) then there is a similarly induced set of coordinates \((\tilde{q}^1, \ldots, \tilde{q}^n, \tilde{v}^1, \ldots, \tilde{v}^n)\) for \( TQ \). Quite clearly, by Lemma 2.2.4, the velocity coordinates are related by
\[
\tilde{v}^i = \frac{\partial q^i}{\partial \tilde{q}^j} v^j, \quad i = 1, \ldots, n. \tag{2.5}
\]
Note that we wish to think of velocity as being an independent coordinate, and so shall steer away from using the symbols \((\tilde{q}^1, \ldots, \tilde{q}^n)\) to denote velocity, unless we actually are dealing with a curve \( t \mapsto (\tilde{q}^1(t), \ldots, \tilde{q}^n(t)) \), in which case \((\tilde{q}^1(t), \ldots, \tilde{q}^n(t))\) have their usual meaning as time derivatives. To denote the velocity coordinates we shall use the symbols \((v^1, \ldots, v^n)\). Note, however, that this is not entirely standard notation, although we feel it to be superior to using \( \dot{q} \) when \( \dot{q} \) is not really what is intended.

Let us see how this notation plays out for our running example.

2.2.6 Example (Example 2.1.5 cont’d) We have \( Q = \mathbb{R}^2 \) with \((U_1, \phi_1)\) the Cartesian coordinate chart and \((U_2, \phi_2)\) the polar coordinate chart. Thus we have coordinates \((x, y, v_x, v_y)\) and \((r, \theta, v_r, v_\theta)\) for \( TQ \). To see how these are related, we use (2.5). Thus we have
\[
v_r = \frac{\partial r}{\partial x} v_x + \frac{\partial r}{\partial y} v_y = \frac{x}{\sqrt{x^2 + y^2}} v_x + \frac{y}{\sqrt{x^2 + y^2}} v_y
\]
and
\[
v_\theta = \frac{\partial \theta}{\partial x} v_x + \frac{\partial \theta}{\partial y} v_y = -\frac{y}{x^2 + y^2} v_x + \frac{x}{x^2 + y^2} v_y.
\]
Notice that this is just as velocities should behave in that we would also compute
\[
\dot{r} = \frac{x}{\sqrt{x^2 + y^2}} \dot{x} + \frac{y}{\sqrt{x^2 + y^2}} \dot{y}
\]
and
\[
\dot{\theta} = -\frac{y}{x^2 + y^2} \dot{x} + \frac{x}{x^2 + y^2} \dot{y}.
\]
It generally does not hurt to think of \( v^i \) as being like \( \dot{q}^i \), at least heuristically. However, when one wishes to be precise, this notation often betrays us. \(\square\)

2.2.2 Vector fields Vector fields are extremely useful objects in many fields of applied mathematics, and they possess many surprising properties that may be used to answer all manner of interesting questions. We will get some exposure to the utility of the vector field in our discussion of mechanics. There are at least two ways in which one can think of a vector field, and we shall start with that suggested by the words “vector field.”

**Vector fields as...well...fields of vectors** Let \( Q \) be a configuration space. A vector field on \( Q \) is a map \( X : Q \to TQ \) with the property that \( X(q) \in T_qQ \). The idea of a vector field is one with which you are undoubtedly familiar. The idea is that \( X \) assigns to each point \( q \in Q \) a point in the tangent space \( T_qQ \), and this is to be thought of as a vector anchored at the point \( q \) (see Figure 2.5).

Let us see how we represent vector fields in coordinates. Choose a chart \((U, \phi)\) for \( Q \) with coordinates \((q^1, \ldots, q^n)\). The vectors \( \{ \frac{\partial}{\partial q^1} \bigg|_q, \ldots, \frac{\partial}{\partial q^n} \bigg|_q \} \) form a basis for \( T_qQ \), and let
2.2 Vector fields, one-forms, and Riemannian metrics

A planar vector field

Figure 2.5

us use these to define $n$ vector fields $\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\}$ on the subset $U$ of $Q$. Then any vector field can be written as

$$X(q) = X^i(q) \frac{\partial}{\partial q^i}$$

for functions $X^1, \ldots, X^n$ on $Q$. Following what we did with tangent vectors, these are the components of $X$ in the given coordinate chart. Of course, the components of a vector field obey the same transformation properties that tangent vectors obey when we change coordinate charts. Note that we will always ask that the components of a vector field be at least continuously differentiable. If the components are not, then we will generally disallow the resulting object from being a vector field.

2.2.7 Example (Example 2.1.5 cont’d) In Example 2.2.3 we looked at tangent vectors on $Q = \mathbb{R}^2$ in Cartesian coordinates and polar coordinates. These may be regarded as vector fields merely by a change of notation. Thus we have a vector field $X$ on $Q$ that is represented as

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

in Cartesian coordinates, and by

$$X = \frac{\partial}{\partial \theta}$$

in polar coordinates. The Cartesian coordinate version of this vector field is the one depicted in Figure 2.5.

Vector fields and ordinary differential equations In the above discussion of vectors, we were merely thinking of vectors as geometric objects that, by definition, obeyed certain properties with respect to coordinate changes. It turns out that vector fields also are in some sense equivalent to something you already know about: ordinary differential equations.

To make this connection, let $X$ be a vector field on $Q$. A differentiable curve $c: [0, T] \to Q$ is an integral curve for $X$ if for every set of coordinates $(q^1, \ldots, q^n)$ for $Q$ in which the curve $c$ has the representation $t \mapsto (q^1(t), \ldots, q^n(t))$, the curve $c$ satisfies

$$\dot{q}^1(t) = X^1(q^1(t), \ldots, q^n(t))$$

$$\vdots$$

$$\dot{q}^n(t) = X^n(q^1(t), \ldots, q^n(t)).$$

Note therefore, that an integral curve for $X$ is determined by its being a solution of an ordinary differential equation in every coordinate chart, with the right-hand side vector of the ordinary differential equation being the components of the vector field in that set of coordinates.

You may be used to trying to solve differential equations when you see them. This is an almost always futile exercise. It works for differential equations that are linear (but see Exercise E2.16), and for a small class of other examples, but when faced with a randomly selected differential equation, you are simply not going to be able to solve it. And even when you can get a closed form solution, it is often not entirely helpful. For example, one can solve the nonlinear pendulum equation in terms of elliptic functions. But so what? Unless you know a lot about elliptic functions, then you are not going to know very much about the behaviour of a pendulum by looking at its closed form solution. So how should we talk about differential equations in a sensible manner? The answer lies in trading off a quantitative description of the dynamics, and to do this, one needs some concepts that may be new. We will not be entirely thorough in our introduction to the qualitative aspects of ordinary equations as they relate to vector fields.

Let $X$ be a vector field on $Q$ and let $q_0 \in Q$. If we are working in a coordinate chart $(U, \phi)$ for which $q \in U$ then, if $q_0 = \phi(q_0) \in \mathbb{R}^n$, we have a unique solution to the initial value problem

$$\dot{q}^1(t) = X^1(q^1(t), \ldots, q^n(t)), \quad q^1(0) = q_0^1$$

$$\vdots$$

$$\dot{q}^n(t) = X^n(q^1(t), \ldots, q^n(t)), \quad q^n(0) = q_0^n.$$  

This implies that there is a unique integral curve $c$ with the property that $c(0) = q_0$. Now, this integral curve may not be extensible for all time, so we let

$$T(X, q_0) = \sup_{t \in \mathbb{R}} \{\text{there exists an integral curve of } X \text{ through } q_0 \text{ defined on } [0, T]\}.$$  

For many well-behaved systems, $T(X, q_0) = \infty$, but there are some innocuous systems for which $T(X, q_0)$ is finite.\footnote{If you have had a decent differential equations course, you will know, for example, that the solutions to the differential equation $\ddot{x} = x^2$ exist only for a finite time.} Now we define

$$D(X) = \{ (t, q) \in \mathbb{R}_+ \times Q \mid t < T(X, q) \}.$$  

Thus for $(t, q) \in D(X)$ there is an integral curve through $q$ that can be defined for at least time $t$. The flow of $X$ is the map $F_X: D(X) \to Q$ defined by $F_X(t, q_0) = q$ if there exists an integral curve $c$ with $c(0) = q_0$ and $c(t) = q$. The idea is that $F_X(t, q_0)$ is where you end up when you start at $q_0$ at time $0$ and go along with the solution of the differential equation for time $t$.

Let us work this out for our example vector field.
2.2.8 Example (Example 2.1.5 cont’d) We again have the two coordinate charts \((U_1, \phi_1)\) and \((U_2, \phi_2)\) that have their previous meanings. We also have the vector field \(X\) that in the coordinate chart \((U_1, \phi_1)\) is given by

\[ X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \]

and in the coordinate chart \((U_2, \phi_2)\) is given by

\[ X = \frac{\partial}{\partial \theta}. \]

The differential equation in the coordinate chart \((U_1, \phi_1)\) is then

\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= x,
\end{align*}
\]

which is linear, and for the initial conditions \((x(0), y(0))\) has the solution

\[
(x(t), y(t)) = (x(0) \cos t - y(0) \sin t, x(0) \sin t + y(0) \cos t).
\]

Note that solutions exist for all time, so for each \((x, y) \in Q\) we have \(T(X, (x, y)) = \infty\). Then the flow is then defined by

\[
F_X(t, (x, y)) = (x \cos t - y \sin t, x \sin t + y \cos t).
\]

Let us look at the same thing in the other coordinate chart. The differential equations in this coordinate chart are

\[
\begin{align*}
\dot{r} &= 0 \\
\dot{\theta} &= 1.
\end{align*}
\]

Note that these equations are no longer linear, but are trivially solved for the initial condition \((r(0), \theta(0))\) as

\[
(r(t), \theta(t)) = (r(0), \theta(0) + t).
\]

One might be tempted to say that from this one we may deduce that \(T(X, (r(0), \theta(0))) = \infty\), but be careful, because this coordinate chart does not cover all of \(Q\). To see in fact verify that all integral curves may be extended for all time, one actually needs to find another chart to extend the integral curves. But we do not need to do this here since we already know from our use of the coordinate chart \((U_1, \phi_1)\) that we may extend all integral curves for infinite time.

2.2.3 One-forms As we have seen, a vector field on \(Q\) is an object defined in a coordinate chart that has \(n\) components if \(n\) is the dimension of \(Q\). We shall shortly see that a “one-form” also has this exact property. This will confuse a newcomer, and many a long-time practitioner has only a shady understanding of the difference between a one-form and a vector field. The fact of the matter is that they live in entirely different spaces. Thus, to get things started, let us look at something that you may be tempted to call a vector field, and show that it is actually not a vector field. Let \(f: Q \to \mathbb{R}\) be a differentiable function and let \((U, \phi)\) be a coordinate chart for \(Q\). We then have the “gradient” of \(f\) that we denote by \(df\). Let us agree not to call this the gradient, but the differential of \(f\). Whatever we call it, it is certainly an object with \(n\) components, and these components are none other than

\[
\frac{\partial f}{\partial q^1}, \ldots, \frac{\partial f}{\partial q^n},
\]

if \((q^1, \ldots, q^n)\) are coordinates in the chart \((U, \phi)\). Is this a vector field? To determine the answer to this question, we suppose that we have another set of overlapping coordinates \((\tilde{q}^1, \ldots, \tilde{q}^n)\) so that the components of the abstract object \(df\) in these new coordinates are

\[
\frac{\partial f}{\partial \tilde{q}^1}, \ldots, \frac{\partial f}{\partial \tilde{q}^n}.
\]

To relate the two different coordinate expressions, we use the chain rule:

\[
\frac{\partial f}{\partial q^i} = \frac{\partial \tilde{q}^j}{\partial q^i} \frac{\partial f}{\partial \tilde{q}^j}.
\]

But this is not how the components of a vector field should transform with respect to coordinate changes, cf. Lemma 2.2.4. Hold on a minute here! The components of a vector field in our indexing convention are supposed to be superscripts. But the index in \(\frac{\partial f}{\partial q^i}\) is a subscript (a superscript in the denominator). So things are really not looking good for \(df\) being a vector field. So what is it? It is a one-form!

Now let’s get serious.

Recall from Appendix A that the dual space \(V^*\) to a vector space \(V\) is the collection of linear maps from \(V\) to \(\mathbb{R}\). The dual space is a vector space with the same dimension as \(V\). It turns out that the tangent spaces \(T_q Q\) to a configuration space \(Q\) are themselves vector spaces. This is clear in coordinates. Let \((U, \phi)\) be a coordinate chart with coordinates \((q^1, \ldots, q^n)\) for \(Q\) and coordinates \((\tilde{q}^1, \ldots, \tilde{q}^n)\) for \(T_q Q\). Then for fixed \(q_0 \in Q\) with \((q^1_0, \ldots, q^n_0) = \phi(q_0)\), the tangent space \(T_{q_0} Q\) is represented by

\[
\{ (\tilde{q}^1_0, \ldots, \tilde{q}^n_0, \tilde{v}^1, \ldots, \tilde{v}^n) \mid (\tilde{v}^1, \ldots, \tilde{v}^n) \in \mathbb{R}^n \}.
\]

The vector space structure on \(T_q Q\) is inherited by the usual vector addition and scalar multiplication on \(\mathbb{R}^n\).

With this background, let us look at the dual space to \(T_q Q\).

2.2.9 Definition The dual space to \(T_q Q\) we denote by \(T_q^* Q\) and call the cotangent space. Following what we did with tangent vectors, let us agree to write a typical point in the cotangent space \(T_q^* Q\) as \(\alpha_q\). An element \(\alpha_q \in T_q^* Q\) is a covector at \(q\). The collection

\[
\bigcup_{q \in Q} T_q^* Q
\]

of all cotangent spaces is called the cotangent bundle of \(Q\) and denoted \(T^* Q\).
where \( \{e^1, \ldots, e^n\} \) is the basis for \((\mathbb{R}^n)^*\) that is dual to the standard basis. Let us denote by \( dq^i|_{q_i} \), \( i = 1, \ldots, n \), these \( n \) covectors. As with tangent vectors, this notation is strange but convenient. Given an arbitrary covector \( \alpha_i \in T^*_q Q \), we may write
\[
\alpha_i = \alpha, dq^i|_{q_i}
\]
for some \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) called the components of \( \alpha \). Note that we are still using the summation convention, but things for covectors look different from what we have for tangent vectors. The indices for the basis covectors are superscripts (they were subscripts for the basis tangent vectors) and the indices for the components of a covector are subscripts (they were superscripts for the components of a tangent vectors).

Now let us observe how the basis covectors and their components react to changes of coordinate. In the following lemma, we let \((q^1, \ldots, q^n)\) be coordinates in a chart \((U_1, \phi_1)\) and \((\tilde{q}^1, \ldots, \tilde{q}^n)\) be coordinates in a chart \((U_2, \phi_2)\).

**2.2.10 Lemma** \( dq_i|_{q_i} = \frac{\partial (\phi_2(q))}{\partial q^i}|_{q_i} dq^j|_{q_j} \), \( i = 1, \ldots, n \).

**Proof** One the intersection \( U = U_1 \cap U_2 \) of the chart domains we have
\[
\eta_i dq^i|_{q_i} = dq_i|_{q_i}
\]
for some numbers \( \eta_1, \ldots, \eta_n \). Let us apply both sides of this equation to the tangent vector \( \frac{\partial}{\partial q^i} \)
\[
\eta_i = dq^j|_{q_j} \left( \frac{\partial}{\partial q^i} \right) = dq^j|_{q_j} \left( \frac{\partial (\phi_2(q))}{\partial q^i} \right) = dq^j|_{q_j} \frac{\partial}{\partial q^j}.
\]

Now, it should be clear that the components of a covector transform as follows.

**2.2.11 Lemma** Let \( \alpha \) be a one-form on \( Q \) and let \((q^1, \ldots, q^n)\) and \((\tilde{q}^1, \ldots, \tilde{q}^n)\) be coordinates for \( Q \). If \( \alpha_1, \ldots, \alpha_n \) are the components for \( \alpha \) in the coordinates \((q^1, \ldots, q^n)\), then
\[
\left( \frac{\partial q^1}{\partial \tilde{q}^1}, \ldots, \frac{\partial q^n}{\partial \tilde{q}^n} \right) \alpha \text{ are the components of } \alpha \text{ in the coordinates } \left( \tilde{q}^1, \ldots, \tilde{q}^n \right).
\]

Now that we understand what we mean by a covector, we can easily define a **one-form** on \( Q \) to be a map \( \alpha : Q \to T^*_q Q \). Just as we did with vector fields, we can write a one-form in coordinates \((q^1, \ldots, q^n)\) as
\[
\alpha = \alpha_i dq^i
\]
where the functions \( \alpha_1, \ldots, \alpha_n \) are the components of \( \alpha \). As with vector fields, we require that the components of a one-form be at least continuously differentiable when represented in coordinates.

Let us look at an example.

**2.2.12 Example** (Example 2.1.5 cont’d) As usual in our running example, we have \( Q = \mathbb{R}^2 \) with two coordinate charts \((U_1, \phi_1)\) and \((U_2, \phi_2)\). First let’s see how the basis one-forms in each coordinate chart are related. We have, by Lemma 2.2.10,
\[
dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos \theta dr - r \sin \theta d\theta.
\]

and
\[
dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin \theta dr + r \cos \theta d\theta.
\]

Now let us consider a one-form that is the differential of a function. We take as our function that defined in the coordinate chart \((U_1, \phi_1)\) to be \( f(x, y) = \frac{1}{2}(x^2 + y^2) \). In the chart \((U_1, \phi_1)\) we have
\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = x dx + y dy.
\]

Note that \( dx \) and \( dy \) are not so called “infinitesimals,” but are linearly independent one-forms!

In polar coordinates we have \( f(r, \theta) = \frac{1}{2}r^2 \), so
\[
df = rdr.
\]

These components ought to be related as by Lemma 2.2.11, so let us check this directly. We compute
\[
\frac{\partial f}{\partial r}(x) + \frac{\partial f}{\partial \theta}(y) = \cos \theta (r \cos \theta) + \sin \theta (r \sin \theta) = r
\]
and
\[
\frac{\partial f}{\partial \theta}(x) + \frac{\partial f}{\partial \theta}(y) = -r \sin \theta (r \cos \theta) + r \cos \theta (r \sin \theta) = 0,
\]
and this is as it should be. \( \square \)

Note that one-forms “eat” vector fields. That is, if we have a vector field \( X \) and a one-form \( \alpha \), then we have a function on \( Q \) defined by
\[
Q \ni q \mapsto \alpha(q)(X(q)) \in \mathbb{R}.
\]

We denote this function by \( \langle \alpha, X \rangle \), or \( \alpha(X) \), or \( \alpha \cdot X \), whichever pleases us. This should not be confusing since whenever you see a one-form and a vector field sitting beside one another, about the only thing they can do is combine to give a function!

**2.2.13 Example** (Example 2.1.5 cont’d) We have the vector field \( X \) defined in Cartesian coordinates by
\[
X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y},
\]
and the one-form defined in Cartesian coordinates by
\[
df = x dx + y dy.
\]

In polar coordinates we have
\[
X = \frac{\partial}{\partial r}, \quad df = rdr.
\]

Thus, in Cartesian coordinate we compute
\[
\langle df; X \rangle = x(-y) + y(x) = 0,
\]
and similarly, in polar coordinates we have
\[
\langle df; X \rangle = r(0) + 0(1) = 0.
\]

Thus when we feed \( X \) to the one-form \( df \) we get zero in this case. In this case we can interpret this as follows: The function \( f \) does not change in the direction of the vector field \( X \). \( \square \)
Just as we can think of the tangent bundle $TQ$ as an object having its own sets of coordinates, we can think of the cotangent bundle $T^*Q$ as such an object as well. The way we do so is entirely analogous with how this is done in the tangent bundle case. Given a coordinate chart $(U, \phi)$ for $Q$ with coordinates $(q^1, \ldots, q^n)$, and given $\bar{q} \in U$, we can write a typical element $\alpha_{\bar{q}} \in T^*_Q \bar{q}$ as

$$\alpha_{\bar{q}} = p_i dq^i |_{\bar{q}}$$

for some $p_1, \ldots, p_n \in \mathbb{R}$. We think of $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ as coordinates for $T^*Q$, which we call the natural coordinates associated with the coordinate chart $(U, \phi)$. The coordinates for the “one-form part” transform like

$$\tilde{p}_i = \frac{\partial q^j}{\partial \tilde{q}^i} \tilde{p}_j, \quad i = 1, \ldots, n. \quad (2.6)$$

This works out as we might expect in our polar coordinate example.

2.2.14 Example (Example 2.1.5 cont’d) We again have $Q = \mathbb{R}^2$ with $(U_1, \phi_1)$ the Cartesian coordinate chart and $(U_2, \phi_2)$ the polar coordinate chart. Thus natural Cartesian coordinates for $T^*Q$ are denoted $(x, y, p_x, p_y)$ and natural polar coordinates are denoted by $(r, \theta, p_r, p_\theta)$. These coordinates are related by

$$p_x = \frac{\partial x}{\partial \tilde{q}^i} \tilde{p}_i + \frac{\partial y}{\partial \tilde{q}^i} \tilde{p}_i = \frac{x}{\sqrt{x^2 + y^2}} p_x + \frac{y}{\sqrt{x^2 + y^2}} p_y$$
$$p_y = \frac{\partial x}{\partial \tilde{q}^i} \tilde{p}_i + \frac{\partial y}{\partial \tilde{q}^i} \tilde{p}_i = -y p_x + x p_y.$$

As we shall see in Section 2.9, these are what are called “conjugate momenta” in the Hamiltonian formalism.

2.2.15 Remark The Examples 2.2.6 and 2.2.14 exhibit some interesting phenomenon which are general. If we write

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{pmatrix} \begin{pmatrix} \tilde{q}_x \\ \tilde{q}_y \end{pmatrix},$$

then we note, by direct calculation, that

$$\begin{pmatrix} \tilde{p}_x \\ \tilde{p}_y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{pmatrix}^{-1} \begin{pmatrix} p_x \\ p_y \end{pmatrix}.$$}

This will generally be the case, and is simply a reflection of the fact that the matrix with components $\frac{\partial q^i}{\partial \tilde{q}^j}$, $i, j = 1, \ldots, n$, is the inverse of the transpose of the matrix with components $\frac{\partial \tilde{q}^i}{\partial q^j}$, $i, j = 1, \ldots, n$. Provided we agree that for $\frac{\partial \tilde{q}^i}{\partial q^j}$ the “up” index is the row index and the “down” index is the column index, and that for $\frac{\partial q^i}{\partial \tilde{q}^j}$ the “up” index is the column index and the “down” index is the row index. This is consistent with the usage if the equations (2.5) and (2.0) if they are to be thought of as matrix-vector multiplication. \hspace{1cm} \square

2.2.4 Riemannian metrics Let us now move briefly on to talking about another important object, particularly in mechanics. A Riemannian metric on a configuration space $Q$ is an assignment of an inner product to each tangent space $T_p Q$. This seems straightforward enough. How do we represent a Riemannian metric in coordinates? Well, let $(q^1, \ldots, q^n)$ be coordinates for a chart $(U, \phi)$ and let $g$ be a Riemannian metric. Since $g(q)$ is an inner product on the vector space $T_p Q$, it will, in particular, take any two vectors in $T_p Q$ and return a number. If we feed $g(q)$ two of the basis vectors from the set $\{\frac{\partial}{\partial q^1}|_{q}, \ldots, \frac{\partial}{\partial q^n}|_{q}\}$, then we may define the $n^2$ numbers $g_{ij}(q)$, $i, j = 1, \ldots, n$, by

$$g_{ij}(q) = g(q)(\frac{\partial}{\partial q^i}|_{q}, \frac{\partial}{\partial q^j}|_{q}).$$

These are the components of $g(q)$ in the coordinate chart $(U, \phi)$. Are these components with respect to a basis? Yes, and let us describe this basis. For $i, j = 1, \ldots, n$ we define a map

$$\frac{dq^i}{\sqrt{|g|}} \otimes \frac{dq^j}{\sqrt{|g|}} : T_p Q \times T_p Q \rightarrow \mathbb{R}$$

by defining it on basis elements by

$$\frac{dq^i}{\sqrt{|g|}} \otimes \frac{dq^j}{\sqrt{|g|}} (\frac{\partial}{\partial q^k}|_{q}), \ell = 1, 2, \ldots, n \quad \text{otherwise}.$$

If we feed $\frac{dq^1}{\sqrt{|g|}} \otimes \frac{dq^1}{\sqrt{|g|}}$ two general vectors, say $u = u^k \frac{\partial}{\partial q^k}|_{q}$ and $v = v^\ell \frac{\partial}{\partial q^\ell}|_{q}$, we declare by linearity that

$$\frac{dq^i}{\sqrt{|g|}} \otimes \frac{dq^j}{\sqrt{|g|}} (u, v) = u^k v^\ell.$$

Now with this notation we claim that $g$ is represented in our set of coordinates by

$$g(q) = g_{ij}(q) \frac{dq^i}{\sqrt{|g|}} \otimes \frac{dq^j}{\sqrt{|g|}}.$$

Indeed, for $u = u^k \frac{\partial}{\partial q^k}|_{q}$ and $v = v^\ell \frac{\partial}{\partial q^\ell}|_{q}$, we have

$$g(q)(u, v) = g_{ij}(q) u^i v^j,$$

and this indeed how the correct formula for an inner product. Let us see how the bases and the components change when we change coordinates. We will not prove what we say here, as it is straightforward along the lines of Lemma 2.2.10 and Lemma 2.2.11. As usual, in the following result we are considering $(q^1, \ldots, q^n)$ as coordinates in a chart $(U_1, \phi_1)$ and $(\tilde{q}^1, \ldots, \tilde{q}^n)$ as coordinates in a chart $(U_2, \phi_2)$.

2.2.16 Lemma \hspace{0.1cm} $\frac{dq^i}{\sqrt{|g|}} \otimes \frac{dq^j}{\sqrt{|g|}} = \frac{\partial q^i}{\partial \tilde{q}^k} \frac{\partial q^j}{\partial \tilde{q}^\ell} \frac{dq^k}{\sqrt{|g|}} \otimes \frac{dq^\ell}{\sqrt{|g|}},$ \hspace{0.1cm} $i, j = 1, \ldots, n.$

The components now change in the predictable manner.

2.2.17 Lemma \hspace{0.1cm} Let $g$ be a Riemannian metric on $Q$ and let $(q^1, \ldots, q^n)$ and $(\tilde{q}^1, \ldots, \tilde{q}^n)$ be coordinates for $Q$. If $\tilde{g}_{ij}$, $i, j = 1, \ldots, n$, are the components for $g$ in the coordinates $(q^1, \ldots, q^n)$, then

$$\frac{\partial q^i}{\partial \tilde{q}^j}, \tilde{g}_{ij} \tilde{q}^k \tilde{q}^\ell \frac{\partial q^j}{\partial \tilde{q}^k} \frac{\partial q^i}{\partial \tilde{q}^\ell}, \quad i, j = 1, \ldots, n,$$
The following example shows that this is all not so bad. In the example, we illustrate how one deals with Riemannian metrics in practice.

2.2.18 Example (Example 2.1.5 cont’d) We again take $Q = \mathbb{R}^2$ with its two coordinate charts $(U_1, \phi_1)$ and $(U_2, \phi_2)$. Let us define a Riemannian metric on $Q$ by defining it in the chart $(U_1, \phi_1)$. We define
\[
g = dx \otimes dx + dy \otimes dy.
\]
This is something you already know about. Indeed, let $X$ and $Y$ be general vector fields given by
\[
X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}, \quad Y = Y_1 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y}.
\]
We then have
\[
g(X, Y) = (dx \otimes dx + dy \otimes dy) \left( X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}, Y_1 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y} \right)
\]
\[
= dx \otimes dx (X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}) + dy \otimes dy (X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y})
\]
\[
= X_1 Y_1 + X_2 Y_2.
\]

The other way to reach the same answer is by using the formula in Lemma 2.2.17. Just as before.

As you can see, it is often more straightforward in practice to use the change of basis to determine the components for a Riemannian metric in a new set of coordinates. □

Associated with a Riemannian metric are two useful pieces of notation. Since $g$ is definite, the matrix with components $g_{ij}, i, j = 1, \ldots, n,$ is invertible at each point in any coordinate system. This allows us to define an invertible map $g^t: TQ \to T^*Q$ by
\[
(g^t(v_q); u_q) = g(v_q, u_q).
\]
One readily checks (see Exercise E2.17) that in a set of coordinates $(q^1, \ldots, q^n)$ that
\[
g^t \left( \frac{\partial}{\partial q^i} \right) = g_{ij} dq^j, \quad i = 1, \ldots, n,
\]
if $g_{ij}, i, j = 1, \ldots, n,$ are the components of $g$ in the given set of coordinates. Therefore, the representation of $g^t$ in coordinates is
\[
g^t(q^1, \ldots, q^n, v^1, \ldots, v^n) = (q^1, \ldots, q^n, g_{ij} v^i, \ldots, g_{nj} v^n).
\]

(2.7)

Note in particular that $g^t$ maps the tangent space $T_q Q$ into the cotangent space $T^*_q Q$. As $g$ is definite, $g^t$ is invertible, and we write its inverse as $g^t: T^*Q \to TQ$. If we let $g^t, i, j = 1, \ldots, n,$ denote the components of the matrix which is the inverse of the matrix with components $g_{ij}, i, j = 1, \ldots, n,$ then
\[
g^t(q^1, \ldots, q^n, p_1, \ldots, p_n) = (q^1, \ldots, q^n, g^{ij} p_i, \ldots, g^{nj} p_n).
\]
We call the maps $g^t$ and $g^t$ the musical isomorphisms.

2.2.19 Example (Example 2.1.5 cont’d) This all works out quite simply in our running example where $Q = \mathbb{R}^2$. We, as previously,
\[
g = dx \otimes dx + dy \otimes dy.
\]

In the Cartesian coordinates, one readily verifies that
\[
g^t(x, y, v_x, v_y) = (x, y, v_x, v_y).
\]
This looks like we are not saying anything, but remember that on the left, $(x, y, v_x, v_y)$ are coordinates for a point in $TQ$, whereas on the right they are coordinates for a point in $T^*Q$. We also clearly have
\[
g^t(x, y, p_x, p_y) = (x, y, p_x, p_y).
\]
Here again, be careful in saying that nothing is happening. On the right, \((x, y, p_x, p_y)\) are coordinates for a point in \(T^*Q\), and on the left they are coordinates for a point in \(TQ\).

This becomes less opaque when we represent \(g^t\) in polar coordinates:

\[
g^t(r, \theta, v_r, v_\theta) = (r, \theta, v_r, r^2v_\theta)
\]

and

\[
g^t(r, \theta, p_r, p_\theta) = (r, \theta, p_r, \frac{1}{r}p_\theta).
\]

Although in Cartesian coordinates the maps \(g^t\) and \(g^t\) are "trivial," they are less so in polar coordinates. However, both are simply coordinate representations for the same thing. □

### 2.3 A variational principle

Many important ideas in mechanics have a variational basis. The calculus of variations is a wide-ranging and complex subject, and we shall only be able to deal with it superficially. Nevertheless, the notion that the laws of nature act in such a way as to extremise some function is an important one.

#### 2.3.1 Lagrangians

Let \(Q\) be a configuration space with tangent bundle \(TQ\). A Lagrangian is a function on \(\mathbb{R} \times TQ\). Thus \(L\) is a function of time, position, and velocity. If \((U, \phi)\) is a coordinate chart with coordinates \((q^1, \ldots, q^n)\) we might write \(L(t, q^1, \ldots, q^n, v^1, \ldots, v^n)\) or \(L(t, q, v)\) for short, if we are working in coordinates.

We will wish to evaluate Lagrangians along curves. To do so, we need to say how to regard the velocity vector along a curve as a coordinate independent object. Let us start by denoting this tangent vector by \(\dot{c}(t)\). But this is, as it surely must be, exactly how a tangent vector should behave. Let us then regard the velocity vector along a curve as a coordinate independent object. Let us start by denoting this tangent vector by \(\dot{c}(t)\). But this is, as it surely must be, exactly how a tangent vector should behave.

#### 2.3.2 Variations

We will be wishing to minimise a certain function over a class of curves. To do this we need to say how to "vary" a curve in just the same way as one needs to be able to vary a function near a point to determine its derivative.

#### 2.3.2.1 Definition

Let \(c: [a, b] \rightarrow Q\) be a curve that is twice continuously differentiable. A variation of \(c\) is a map \(\sigma: [-\epsilon, \epsilon] \times [a, b] \rightarrow Q\) with the properties

(i) \(\sigma(0, t) = c(t)\),

(ii) \(\sigma(s, a) = c(a)\), and

(iii) \(\sigma(s, b) = c(b)\).

The infinitesimal variation associated with a variation \(\sigma\) is the vector field defined at points along \(c\) by

\[
\delta\sigma(t) = \frac{d}{ds}\bigg|_{s=0} \sigma(s, t) \in T_{c(t)}Q.
\]

The idea is that a variation of \(c\) is a "wiggling" of \(c\), and an infinitesimal variation corresponding to a certain manner of wiggling is a measure of the wiggle for small values of the "wiggle parameter." Some intuition is as seen in Figure 2.7. Note that the endpoints of a variation, as we have defined it, remain stationary, so the infinitesimal variations vanish at the endpoints.

#### 2.3.3 Statement of the variational problem and Euler's necessary condition

We let \(Q\) be a configuration space, and let \(a < b \in \mathbb{R}\) with \(q_a, q_b \in Q\). We let

\[
C^2([a, b], q_a, q_b) = \{c: [a, b] \rightarrow Q\} \quad c(a) = q_a, \quad c(b) = q_b, \quad \text{and} \quad c \text{ is twice continuously differentiable}\}
\]

be the collection of twice continuously differentiable curves that are defined on \([a, b]\) and that start at \(q_a\) and end at \(q_b\). Now we suppose that we have a Lagrangian on \(Q\) and define a function \(J_L: C^2([a, b], q_a, q_b) \rightarrow \mathbb{R}\) by

\[
J_L(c) = \int_a^b L(t, c(t)) dt,
\]

where \(t \mapsto c'(t)\) is the tangent vector field of \(c\). A curve \(c_0 \in C^2([a, b], q_a, q_b)\) minimises \(J_L\) if \(J_L(c_0) \leq J_L(c)\) for every \(c \in C^2([a, b], q_a, q_b)\).

The following condition gives what might be regarded as a first-order necessary condition for a curve \(c \in C^2([a, b], q_a, q_b)\) to be a minimiser for \(J_L\). Thus it should be thought of as being analogous to the first derivative condition for determining the minimum of a function.
2.3.2 Theorem If a curve $c \in C^2([a, b], q_\alpha, q_\beta)$ minimises $J_\alpha$ then for any chart $(U, \phi)$ with the property that the image of $c$ intersects $U$, the representation $t \mapsto q(t)$ of $c$ in that chart must satisfy

$$ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0, \quad i = 1, \ldots, n, $$

at $(t, q(t), \dot{q}(t))$ for each $t$ with the property that $c(t) \in U$.

Proof Let us first show that if $c$ is a minimiser, then any subarc of $c$ is also a minimiser. Thus let $t_1, t_2 \in [a, b]$ satisfy $a \leq t_1 < t_2 \leq b$, and define a curve $\tilde{c} : [t_1, t_2] \to Q$ by $\tilde{c}(t) = c(t)$ (i.e., $\tilde{c}$ is the restriction of $c$ to $[t_1, t_2]$). With $t_1$ and $t_2$ as defined we can also define the set of curves $C^2([t_1, t_2], c(t_1), c(t_2))$ analogously to our definition of $C^2([a, b], q_\alpha, q_\beta)$. We also define a function $\tilde{J}_\alpha$ on the set of curves $C^2([t_1, t_2], c(t_1), c(t_2))$ by

$$ \tilde{J}_\alpha(\tilde{c}) = \int_{t_1}^{t_2} L(t, \dot{c}(t)) \, dt. $$

We claim that if $c$ is a minimiser for $J_\alpha$, then $\tilde{c}$ is a minimiser for $\tilde{J}_\alpha$. Indeed, if $\tilde{c}$ were not a minimiser for $\tilde{J}_\alpha$, this could imply the existence of a curve $\hat{c}_1 \in C^2([t_1, t_2], c(t_1), c(t_2))$ with the property that $\tilde{J}_\alpha(\hat{c}_1) < \tilde{J}_\alpha(\tilde{c})$. Now define a curve $c_1 : [a, b] \to Q$ by

$$ c_1(t) = \begin{cases} c(t), & t \in [a, t_1] \\ \hat{c}_1(t), & t \in [t_1, t_2] \\ c(t), & t \in [t_2, b] \end{cases} $$

We then have

$$ J_\alpha(c_1) = \int_a^{t_1} L(t, \dot{c}(t)) \, dt + \int_{t_1}^{t_2} L(t, \hat{c}_1'(t)) \, dt + \int_{t_2}^{b} L(t, c(t)) \, dt < \int_a^{t_1} L(t, \dot{c}(t)) \, dt + \int_{t_1}^{t_2} L(t, c(t)) \, dt + \int_{t_2}^{b} L(t, c(t)) \, dt = \int_a^{b} L(t, c(t)) \, dt = J_\alpha(c), $$

This would complete our assertion that the restriction of $c$ to any subinterval of $[a, b]$ is itself a minimiser, but for the fact that the curve $c_1$ may not be twice continuously differentiable, as it may fail to be differentiable at $t_1$ and $t_2$. However, one may show (see Exercise E2.18) that it is possible to modify $c_1$ slightly to a curve $\tilde{c}_1$ so that $\tilde{c}_1$ is twice continuously differentiable on $[a, b]$, and so that $|J_\alpha(\tilde{c}_1) - J_\alpha(c_1)| < \epsilon$ for any $\epsilon > 0$. In this way, one can ensure that $J_\alpha(\tilde{c}_1) < J_\alpha(c_1)$, contradicting the fact that $c$ is a minimiser.

The above argument says that $c$ is a minimiser along every subinterval of $[a, b]$ if it is a minimiser along the entire interval. Therefore, if $(U, \phi)$ is a chart so that $c(t) \in U$ for some $t \in [a, b]$, if $c$ is a minimiser on $[a, b]$, it is also a minimiser for the curve restricted to an interval $[t_1, t_2]$ with the property that $c(t) \in U$ for $t \in [t_1, t_2]$. The upshot is that without loss of generality, we may suppose that the curve $c$ lies in the domain of a chart $(U, \phi)$, and we do this for the remainder of the proof.

Let $\sigma$ be a variation of $c$ and let $c_{\sigma, s} \in C^2([a, b], q_\alpha, q_\beta)$ be defined by $c_{\sigma, s}(t) = \sigma(s, t)$. Consider the function $f_s$ defined on $[-\epsilon, \epsilon]$ by $f_s(s) = J_\alpha(c_{\sigma, s})$. If $c$ is a minimiser for $J_\alpha$ then $s = 0$ should be a minimum of $f_s$ for every variation $\sigma$ of $c$. We let $q_{\sigma, s}$ be the coordinate representative of $c_{\sigma, s}$. We have

$$ 0 = \frac{d}{ds}_{|s=0} f_s(s) = \frac{d}{ds}_{|s=0} \int_a^b L(t, q_{\sigma, s}(t), q_{\sigma, s}'(t)) \, dt $$

$$ = \int_a^b \left( \frac{\partial L}{\partial q_{\sigma, s}} \frac{\partial q_{\sigma, s}}{\partial s} \right) \, dt + \frac{\partial L}{\partial q_{\sigma, s}'} \frac{\partial q_{\sigma, s}'}{\partial s} \bigg|_{s=0} \, dt. $$

Now note that

$$ \frac{\partial q_{\sigma, s}'}{\partial s} \bigg|_{s=0} = \frac{d}{ds}_{|s=0} \frac{\partial q_{\sigma, s}}{\partial s} = \frac{d}{ds}_{|s=0} \frac{\partial q_{\sigma, s}}{\partial s} \bigg|_{s=0} $$

Therefore

$$ 0 = \frac{d}{ds}_{|s=0} f_s(s) = \int_a^b \left( \frac{\partial L}{\partial q_{\sigma, s}} \frac{\partial q_{\sigma, s}}{\partial s} \right) \, dt + \frac{\partial L}{\partial q_{\sigma, s}'} \frac{\partial q_{\sigma, s}'}{\partial s} \bigg|_{s=0} \, dt $$

$$ = \int_a^b \left( \frac{\partial L}{\partial q_{\sigma, s}} \frac{\partial q_{\sigma, s}}{\partial s} \right) \, dt + \frac{\partial L}{\partial q_{\sigma, s}'} \frac{\partial q_{\sigma, s}'}{\partial s} \bigg|_{s=0} \, dt $$

where in the next to last step we have used integration by parts, and in the last step we have used the fact that an infinitesimal variation vanishes at the endpoints. Since

$$ \int_a^b \left( \frac{\partial L}{\partial q_{\sigma, s}} \frac{\partial q_{\sigma, s}}{\partial s} \right) \, dt + \frac{\partial L}{\partial q_{\sigma, s}'} \frac{\partial q_{\sigma, s}'}{\partial s} \bigg|_{s=0} \, dt = 0 $$

for every variation, this means that $\frac{\partial q_{\sigma, s}'}{\partial s} \bigg|_{s=0}$ can be arbitrary, and so the only way for the integral to vanish is for

$$ \frac{\partial L}{\partial q_{\sigma, s}'} \bigg|_{s=0} = 0 $$

to be zero when evaluated at $(q, q(t), \dot{q}(t))$.

The equations

$$ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0, \quad i = 1, \ldots, n, $$

are called the Euler-Lagrange equations, and they are what we will be primarily concerned with in the remainder of these notes. Though innocent enough in appearance, they hide an enormous amount of beautiful structure. Note that the Euler-Lagrange equations are only necessary conditions. A solution to the Euler-Lagrange equations may or may not be an actual minimiser. The common language is to call a curve in $C^2([q_\alpha, q_\beta], [a, b])$ an extremal when it satisfies the Euler-Lagrange equations. This reflects the fact that the Euler-Lagrange equations are essentially analogous to the first derivative conditions in calculus.

Let’s look at a very simple example.

2.3.3 Example (Example 2.1.5 cont’d) We work again with $Q = \mathbb{R}^2$ and its two coordinate charts $(U_1, \phi_1)$ and $(U_2, \phi_2)$. We define a Lagrangian on $Q$ by defining it in the coordinate chart $(U_1, \phi_1)$ to be

$$ L(t, x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2). $$
Suppose that we have a curve \( t \rightarrow (x(t), y(t)) \) that is a minimiser for \( J_L \) for some interval and with some endpoints. Then the Euler-Lagrange equations say that this curve must satisfy
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} = \dot{v}_x(t) = \ddot{x}(t) = 0
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial y} = \dot{v}_y(t) = \ddot{y}(t) = 0.
\]
Thus the Euler-Lagrange equations are the simple second-order differential equations
\[
\ddot{x}(t) = 0, \quad \ddot{y}(t) = 0,
\]
and these have solution \( x(t) = x(0) + \dot{x}(0)t \) and \( y(t) = y(0) + \dot{y}(0)t \). These describe straight lines in the plane \( Q = \mathbb{R}^2 \).

Now let us look at the same example in the polar coordinate chart \((U_2, \phi_2)\). First we need to determine the Lagrangian in these coordinates. Using the transformation rule (2.5) we have
\[
v_x = \frac{\partial x}{\partial r} v_r + \frac{\partial x}{\partial \theta} v_\theta = \cos \theta v_r - r \sin \theta v_\theta
\]
\[
v_y = \frac{\partial y}{\partial r} v_r + \frac{\partial y}{\partial \theta} v_\theta = \sin \theta v_r + r \cos \theta v_\theta.
\]

With these relations we readily compute
\[
v_r^2 + v_\theta^2 = v_x^2 + r^2 v_\theta^2
\]
so that
\[
L = \frac{1}{2}(v_r^2 + r^2 v_\theta^2).
\]
The Euler-Lagrange equations are then
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_r} \right) - \frac{\partial L}{\partial r} = \dot{v}_r(t) - r(t)v_\theta^2(t) = \dot{v}_r(t) - r(t)\dot{\theta}^2(t) = 0
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v_\theta} \right) - \frac{\partial L}{\partial \theta} = \dot{v}_\theta(t) = \ddot{\theta}(t) = 0.
\]
That is, the Euler-Lagrange equations are the two coupled second-order differential equations
\[
\dot{v}_r(t) - r(t)\dot{\theta}^2(t) = 0, \quad r^2(t)\ddot{\theta}(t) + 2r(t)\dot{r}(t)\dot{\theta}(t) = 0.
\]
These equations are less amenable to solution by inspection than their Cartesian coordinate counterparts. However, since we know the Cartesian solutions and the change of coordinate formulae, we may in principal write the solutions to the polar coordinate equations. However, to do so is straightforward, and not a very interesting display of symbology.

### 2.3.4 The Euler-Lagrange equations and changes of coordinate

In the previous example, we wrote the Euler-Lagrange equations in two different sets of coordinates, and somehow we think that the two sets of equations should represent the same thing. Let us explore exactly what is meant by this. First of all, we remark that the problem of minimising \( J_L \) over curves in \( C^2([a, b], q_e, q_b) \) obviously does not depend upon any choice of coordinates. Now, if the problem \( J_L \) is not dependent on coordinate chart, then so should be any statements relating to its solution. Thus we should hope that the Euler-Lagrange equations are in some sense coordinate invariant. Indeed they are, and in just the way of the following result.

### 2.3.6 Example (Example 2.1.5 cont’d)

We have, as usual, \( Q = \mathbb{R}^2 \) with its Cartesian and polar coordinate charts. In Example 2.3.3 we derived the Euler-Lagrange equations for a particular Lagrangian in both sets of coordinates. Let us verify that these equations are indeed related as in Proposition 2.3.4.

In Cartesian coordinates the Euler-Lagrange equations were
\[
\ddot{x} = 0, \quad \ddot{y} = 0,
\]
and in polar coordinates we had
\[
\ddot{r} - r\dot{\theta}^2 = 0, \quad r^2 \ddot{\theta} + 2r\dot{r}\dot{\theta} = 0.
\]
We compute
\[ \frac{\partial}{\partial \theta} \left[ \frac{dL}{dt} \right] = \frac{\partial}{\partial \theta} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \dot{\theta} \cos \theta - \dot{\theta} \theta + \cos \theta \sin \theta \]

Similarly we compute
\[ \frac{\partial}{\partial \theta} \left[ \frac{dL}{dt} \right] = -\dot{\theta} \sin \theta - \dot{\theta} \theta + \cos \theta \sin \theta \]

Thus the components do indeed transform as they ought to, although a direct verification of this, even in a simple example, is a not entirely pleasant task.

## 2.4 Simple mechanical systems

“Simple” mechanical systems are not so named because they behave in a particularly simple manner. The usage of the word simple here has a precise meaning, and the fact of the matter is that the vast majority of mechanical systems one encounters are “simple” by our usage of the word. Simple mechanical systems are characterised by having a special sort of Lagrangian.

### 2.4.1 Kinetic energy

The most important feature of a simple mechanical system is that it has kinetic energy. This means something very particular in the terminology we introduced in Section 2.2.4. Let \( g \) be a Riemannian metric on \( Q \). The kinetic energy associated with \( g \) is the function \( K \) on \( TQ \) defined by \( K(v) = \frac{1}{2} g(v, v) \). Note that we need a Riemannian metric to define kinetic energy! This is why we introduced the concept of a Riemannian metric. However, when working with an example, one can often readily determine by elementary methods its kinetic energy in a set of coordinates \((q^1, \ldots, q^n)\), without knowing what is the Riemannian metric in these coordinates. But the fact of the matter is that the Riemannian metric is there. To recover it proceed as follows. The kinetic energy you derive will be a function \( K(q^1, \ldots, q^n, v^1, \ldots, v^n) \). The components \( g_{ij} \), \( i, j = 1, \ldots, n \), of the corresponding Riemannian metric are then determined by

\[ g_{ij} = \frac{\partial^2 K}{\partial v^i \partial v^j}, \quad i, j = 1, \ldots, n. \]

Note that if you apply this formula and the components for \( g \) turn out to involve the velocities, then you have either made a mistake, or the system you are dealing with is not a simple mechanical system.

Let us see how this works in our simple example.

### 2.4.2 Remarks

1. Since we have a 1-1 correspondence between the concept of kinetic energy and a Riemannian metric, let us agree to use the latter as basic, since, as we shall see, a Riemannian metric possesses many interesting properties.

2. In very simple examples like the previous one, one can see that it is a bit overkill to formally compute \( \frac{\partial^2 K}{\partial v^i \partial v^j} \) in order to determine the components of the Riemannian metric; one can simply look at the expression for kinetic energy and write down the corresponding metric components. However, for more complicated system formed on multiple components, one often will write the kinetic energy of each component, then taking the total kinetic energy to be the sum of the component energies. In this case, the formula \( g_{ij} = \frac{\partial^2 K}{\partial v^i \partial v^j} \) can be used, perhaps in a symbolic manipulation language, to produce the metric components with relative ease.

### 2.4.3 Example (Example 2.15 cont’d)

Let us resume talking about a mass \( m \) moving about in the plane. Now let us suppose that the plane is oriented in such a way that the force of gravity acts in the \( y \)-direction in Cartesian coordinates. In this case, our elementary
physics tells us that potential energy should be a “storage function” for the energy that the system has when it is at a given height. This, of course, requires us to declare some height as being the zero height, where the body has no gravitational potential energy. This choice is arbitrary, so let us choose it to be \( y_0 \). The gravitational potential energy is then \( V = ma_\theta (r \sin \theta - r_0 \sin \theta_0) \).

Let us see what this function looks like in polar coordinates. Define \( r_0, \theta_0 \in \mathbb{R} + \) and \( \theta_0 \in (-\pi, \pi) \) by \( y_0 = r_0 \sin \theta_0 \). This supposes that \( y_0 \neq 0 \). We then have

\[ y - y_0 = r \sin \theta - r_0 \sin \theta_0, \]

so the gravitational potential in polar coordinates is

\[ V = ma_\theta (r \sin \theta - r_0 \sin \theta_0). \]

Note that, even in the Newtonian world, this is really only an approximation of the gravitational force (see Exercise E2.24).

Although the concept of potential energy is benign enough, there is actually a reason why it is a wee bit subtle, and we will touch on that soon.

### 2.4.3 The Euler-Lagrange equations for simple mechanical systems

We have now the data to say what we mean by a simple mechanical system.

#### 2.4.4 Definition

A **simple mechanical system** is a triple \((Q, g, V)\) where

- (i) \( Q \) is a configuration space,
- (ii) \( g \) is a Riemannian metric on \( Q \), and
- (iii) \( V \) is a potential energy function on \( Q \).

The **Lagrangian** associated to a simple mechanical system \((Q, g, V)\) is the function on \( TQ \) defined by

\[ L(q, \dot{q}) = \frac{1}{2}g(v_q, v_q) - V(q). \]

One might well ask, “Why use a minus sign on the potential?” Indeed, the “natural” quantity seems to be the total energy \( \frac{1}{2}g(v_q, v_q) + V(q) \). The reasons for using the minus sign cannot be explicated in a coherent manner until we look at the Lagrange-d’Alembert principle in Section 2.5. For now, let us just say that we use it because it works.

Let us look at the Euler-Lagrange equations for simple mechanical systems. To do so, given a potential function \( V \) on \( Q \), define a vector field \( \text{grad} V \) on \( Q \) by

\[ \text{grad} V(q) = g^j (\partial V/\partial q^j). \]

Note that since \( dV \) is a one-form, \( \text{grad} V \) is indeed a vector field as claimed. In coordinates we have

\[ \text{grad} V = g^j \partial V/\partial q^j. \]

Now we may state the following result.

### 2.4.5 Proposition

Let \((Q, g, V)\) be a simple mechanical system with associated **Lagrangian** \( L \). Let \( c: I \rightarrow Q \) be a curve which is represented by \( t \mapsto (q^1(t), \ldots, q^n(t)) \) in a coordinate chart \((U, \phi)\). The following statements are equivalent:

- (i) \( t \mapsto (q^1(t), \ldots, q^n(t)) \) satisfies the Euler-Lagrange equations for the Lagrangian \( L \);
- (ii) \( t \mapsto (q^1(t), \ldots, q^n(t)) \) satisfies the second-order differential equation

\[ \ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = -\left(\text{grad} V\right)^i, \quad i = 1, \ldots, n, \]

where \( \Gamma^i_{jk}, i, j, k = 1, \ldots, n, \) are functions of \( q \) defined by

\[ \Gamma^i_{jk} = \frac{1}{2} g^i_{lk} \left( \frac{\partial g_{jk}}{\partial q^l} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jl}}{\partial q^k} \right). \]

**Proof** In coordinates we have

\[ L = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V(q). \]

We therefore have

\[ \frac{\partial L}{\partial \dot{q}^j} = g_{ij} \dot{q}^i, \quad \frac{\partial L}{\partial q^j} = \frac{1}{2} \frac{\partial g_{ij}}{\partial q^k} \dot{q}^k \dot{q}^j - \frac{\partial V}{\partial q^j}. \]

Therefore the Euler-Lagrange equations are

\[ g_{ij} \ddot{q}^j + \left( \frac{\partial g_{ij}}{\partial q^k} - \frac{1}{2} \frac{\partial g_{kj}}{\partial q^i} \right) \dot{q}^k \dot{q}^j + \frac{\partial V}{\partial q^j} = 0. \]

Since the expression \( \dot{q}^k \dot{q}^j \) is symmetric with respect to transposing the indices \( j \) and \( k \), only the symmetric part of the expression

\[ A_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial q^k} - \frac{1}{2} \frac{\partial g_{jk}}{\partial q^i} \]

with respect to the indices \( j \) and \( k \) will contribute to the expression

\[ \left( \frac{\partial g_{ij}}{\partial q^k} - \frac{1}{2} \frac{\partial g_{kj}}{\partial q^i} \right) \dot{q}^k \dot{q}^j. \]

But the symmetric part of \( A_{ijk} \) with respect to \( j \) and \( k \) is

\[ \frac{1}{2} (A_{ijk} + A_{kji}) = \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{kj}}{\partial q^i} \right). \]

The result now follows by multiplying (2.8) by \( g^{ij} \).

We shall see in the next section the meaning of the functions \( \Gamma^i_{jk}, i, j, k = 1, \ldots, n \). They represent a very important object in the study of simple mechanical systems.

Let us see how the description of the Euler-Lagrange equations plays out for our running example.

### 2.4.6 Example

(Example 2.15 cont’d) We take as our simple mechanical system \((Q, g, V)\) where \( g \) is the Riemannian metric of Example 2.4.1 and \( V \) is the potential function of Example 2.4.3.
In Cartesian coordinates, the resulting Lagrangian on $TQ$ is given by

$$L(x, y, v_x, v_y) = \frac{1}{2} m (v_x^2 + v_y^2) - ma_y (y - y_0).$$

The Euler-Lagrange equations are then readily computed to be

$$m\dddot{x} = 0, \quad m\dddot{y} + ma_y = 0.$$  \hspace{1cm} (2.9)

Given that $g = m(dx \otimes dx + dy \otimes dy)$ in these coordinates, one readily determines that the functions $\Gamma^i_{jk}$ are zero for all $i, j, k = 1, 2$. One also sees that

$$\text{grad} V = a_y \frac{\partial}{\partial y},$$

Therefore the equations of part (ii) of Proposition 2.4.5 read

$$\ddot{x} = 0, \quad \ddot{y} = -a_y,$$

and these equations are obviously equivalent to the equations (2.9).

Now let us also check that things work out in polar coordinates. In these coordinates we have

$$L(r, \theta, v_r, v_\theta) = \frac{1}{2} m (v_r^2 + r^2 v_\theta^2) - ma_y (r \sin \theta - r_0 \sin \theta_0).$$

The Euler-Lagrange equations are then

$$m\dddot{r} - mr\dddot{\theta}^2 + ma_y \sin \theta = 0, \quad mra_r \dddot{\theta} + 2mrt \dot{\theta} + ma_r \cos \theta = 0.$$  \hspace{1cm} (2.10)

To check the equations of part (ii) of Proposition 2.4.5, we note that

$$g = m(dr \otimes dr + r^2 d\theta \otimes d\theta),$$

and a simple, slightly tedious, computation gives

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} r,$$

with the remaining $\Gamma$’s being zero. One also computes

$$\text{grad} V = a_y \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} a_y \cos \theta \frac{\partial}{\partial \theta}.$$

Therefore the equations of part (ii) of Proposition 2.4.5 are

$$\ddot{r} - r \dddot{\theta}^2 = -a_y \sin \theta, \quad \dddot{\theta} + \frac{2}{r} \dot{\theta} = -\frac{1}{r} a_y \cos \theta.$$

These equations are clearly equivalent to the equations (2.10).

Note that the equations are independent of the datum $y_0$ from which potential energy was measured.
which one may associate with the Riemannian metric defining the kinetic energy. Let us look
They are the Christoffel symbols of an affine connection, called the Levi-Civita connection,
2.4.9 Theorem Let $\mathfrak{X}, \mathfrak{Y}$ be a Riemannian metric on a configuration space
2.4.10 Theorem Let $(Q, g, V)$ be a simple mechanical system with associated Lagrangian L.
2.4.11 Example (Example 2.1.5 cont’d) The $Q = \mathbb{R}^2$ example we have been using, although
where we now acknowledge explicitly the dependence of the $\Gamma$’s on $g$. Clearly, there is
nothing preventing us from talking about this equation when we do not use the Christoffel symbols
for the Levi-Civita connection, but rather Christoffel symbols for a general affine connection.
To this end, let $\nabla$ be a general affine connection on $Q$. A curve $c: I \to Q$ is called a
geodesic for $\nabla$ when for any $t_0 \in I$ and any coordinate chart $(U, \phi)$ around $c(t_0)$
with coordinates $(q^1, \ldots, q^n)$, the coordinate representation $t \mapsto (q^1(t), \ldots, q^n(t))$
satisfies the differential equation
$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0, \quad i = 1, \ldots, n,$$
just as claimed.

Now we see that $\Gamma$’s may come up as claimed, but it is still not clear how the $\Gamma$’s of
Proposition 2.4.5 are related to the Christoffel symbols. To see how this relationship is
established, we will associate with a Riemannian metric, i.e., with kinetic energy, a unique
affine connection. In order to carry this out, we need to have at hand the concept of a
Lie bracket between two vector fields $X$ and $Y$ on $Q$. This is defined to be the vector field
$[X, Y]$ on $Q$ which, in any set of coordinates $(q^1, \ldots, q^n)$ is given by
$$[X, Y] = \left( \frac{\partial Y^j}{\partial q^i} X^i - \frac{\partial X^j}{\partial q^i} Y^i \right) \frac{\partial}{\partial q^j}.$$  \hfill (2.12)
Of course, for this definition to make sense, one must check that this definition does not
depend on coordinates, and we leave this straightforward calculation to the reader (Exercise E2.15).
The Lie bracket is an extremely important object in differential geometry, and we
will put it to rather pedestrian usage here in our discussion of affine connections. However,
we consider a curve $c: [a, b] \to Q$ and any coordinate chart $(U, \phi)$ around $c(t_0)$
with coordinates $(q^1, \ldots, q^n)$, the coordinate representation $t \mapsto (q^1(t), \ldots, q^n(t))$
satisfies the differential equation
$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0, \quad i = 1, \ldots, n,$$
where $\Gamma^i_{jk}$, $i, j, k = 1, \ldots, n$, are the Christoffel symbols of $\nabla$ in the coordinate chart $(U, \phi)$.
Obviously, for this to make sense, the definition needs to be shown to be independent of
coordinates. We leave this only slightly tedious computation to the reader (Exercise E2.26).
Note, however, that each of the terms in the geodesic equations are not coordinate independent!
Thus one should not write $\ddot{q}$ as a tangent vector, since it does not obey a transformation law
like a tangent vector should! To talk about acceleration as a vector one genuinely needs an
affine connection. In this case, the notation one uses for the quantity which in coordinates
is $\ddot{q} + \Gamma^i_{jk} \dot{q}^j \dot{q}^k$ is $\nabla_{\partial c(t)} \dot{c}(t)$, and a geodesic is a curve $c$ for which $\nabla_{\partial c(t)} \dot{c}(t) = 0$.
We may now offer the following coordinate independent restatement of Proposition 2.4.5.
\begin{align*}
\nabla_{\partial c(t)} \dot{c}(t) &= -\nabla V(c(t)).
\end{align*}
Let us see how this looks in our running example.

2.4.11 Example (Example 2.1.5 cont’d) The $Q = \mathbb{R}^2$ example we have been using, although
simple, is ample to illustrate why one cannot regard acceleration as a tangent vector. On $Q$
we consider a curve $c: \mathbb{R} \to Q$ defined in Cartesian coordinates by
$$c(t) = (\cos t, \sin t).$$
Thus $c$ describes a circle. In Cartesian coordinates we then have $(x(t), y(t)) = (\cos t, \sin t)$
so that
$$(\dot{x}(t), \dot{y}(t)) = (-\sin t, \cos t), \quad (\ddot{x}(t), \ddot{y}(t)) = (-\cos t, -\sin t).$$
In polar coordinates the curve is represented by $(r(t), \theta(t)) = (1, t)$, but note that $r(t)$ and
$\theta(t)$ are only defined for $t \in (-\pi, \pi)$. We readily compute
$$(\dot{r}(t), \dot{\theta}(t)) = (0, 1), \quad (\ddot{r}(t), \ddot{\theta}(t)) = (0, 0).$$
Therefore the acceleration along the curve is zero in polar coordinates, but nonzero in polar
coordinates. This precludes acceleration from being a tangent vector since it cannot be both
zero and nonzero.
Now let us look at the acceleration represented by the Levi-Civita connection corresponding
to the Riemannian metric $g$ from Example 2.4.1. In Cartesian coordinates, the
Christoffel symbols are zero so we have
$$\nabla_{\partial c(t)} \dot{c}(t) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \cos t \frac{\partial}{\partial x} - \sin t \frac{\partial}{\partial y}. $$
In polar coordinates, the Christoffel symbols were computed in Example 2.4.1, and using these we have
\[ \tilde{\nabla}_{c(t)}c'(t) = (\tilde{r} - r\tilde{\theta}) \frac{\partial}{\partial \tilde{r}} + (\tilde{\theta} + \frac{\tilde{\theta}}{2} \tilde{\theta}) \frac{\partial}{\partial \tilde{\theta}} = -\frac{\partial}{\partial \tilde{\theta}}. \]

Thus in both Cartesian and polar coordinates, the quantity \( \tilde{\nabla}_{c(t)}c'(t) \) is a vector of “unit length” (if \( m = 1 \), remove the quotes) pointing towards the \( \nabla \) origin (see Figure 2.8). The

\[ F(t, q, v) = F_i(t, q, v) dq^i, \]

for some function \( F_1, \ldots, F^n \) of time, configuration, and velocity, called the components
of the force \( F \).

The Lagrange-d’Alembert principle, often simply called “d’Alembert’s principle,” tells us how a force \( F \) should appear in the Euler-Lagrange equations.

2.5.1 Definition Let \( L \) be a Lagrangian on a configuration space \( Q \) and let \( F \) be a force. A curve \( c: [a, b] \to Q \) satisfies the Lagrange-d’Alembert principle for the force \( F \) if for every variation \( \sigma: [-\epsilon, \epsilon] \times [a, b] \to Q \) we have
\[ \frac{d}{ds} \bigg|_{s=0} \int_{a}^{b} L(t, q(s), \dot{q}(s)) \, dt + \int_{a}^{b} \langle F(t, c'(t)); 0 \rangle \, dt = 0. \]

The idea behind this definition is that when the external forces are not present, they make no contribution in the second term in Definition 2.5.1. However, when they are present, they do work, and this is evidently properly accounted for in the variational formulation which we call the Lagrange-d’Alembert principle.

Let us see what are the implications of the Lagrange-d’Alembert principle for the equations describing the motion of a Lagrangian system in the presence of forces.

2.5.2 Proposition Let \( L \) be a Lagrangian on \( Q \) with \( F \) a force on \( Q \). A curve \( c: [a, b] \to Q \) satisfies the Lagrange-d’Alembert principle if and only if for any coordinate chart \((U, \phi)\) which intersects the image of \( c \), the coordinate representation \( t \mapsto (\phi^1(t), \ldots, \phi^n(t)) \) satisfies
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i, \quad i = 1, \ldots, n, \tag{2.13} \]

where \( F_1, \ldots, F_n \) are the components of \( F \). The equations (2.13) are the forced Euler-Lagrange equations.

Proof Fix a coordinate chart \((U, \phi)\) which intersects the image of \( c \). By considering variations of \( c \) whose infinitesimal variation vanishes outside a coordinate chart, we may suppose that \( c(t) \in U \) for each \( t \). Let \( \sigma \) be a variation of \( c \) with \( \sigma_{s}, \sigma_{s} \) the curve defined by \( \sigma_{s}(t) = \sigma(s, t) \). Denote the coordinate representation of \( \sigma_{s} \) by \( q_{s, \sigma} \). Then, by following the calculations of Theorem 2.3.2, we arrive at
\[ \int_{a}^{b} \left( \frac{\partial L}{\partial \dot{q}^i} \frac{dq_{s, \sigma}(t)}{ds} \right)_{s=0} - \frac{d}{dt} \frac{\partial L}{\partial q^i} + F_i \frac{dq_{s, \sigma}(t)}{ds} \right)_{s=0} \, dt = 0. \]

Since this must hold for every variation, the result follows.
Thus the manner in which one adds forces to the Euler-Lagrange equations, governed by the Lagrange-d’Alembert principle, is very simple: one merely sticks the components of the force on the right-hand side of the Euler-Lagrange equations. The question of why this is the right thing to do often results in a pointless metaphysical discussion. The real reason is, “Because it works,” meaning that the Lagrange-d’Alembert principle agrees with measurement to a tolerable degree. Mathematical descriptions of our physical world must be a slave to empirical data.

2.5.2 Potential forces A potential force is one resulting from potential energy. More usefully, given a potential function $V$ on $Q$, its potential force is the force given by $F(t, x_0) = -dV(q)$. In particular, potential forces are independent of time and velocity. However, potential forces are more than simply independent of time and velocity. They have a useful property with respect to the “work” done by a mechanical system. We need to define work. One normally takes as the definition of work “force times distance.” This is fine for defining work done on straight line paths in Euclidean space. However, our setting is more general than this in several ways. First, we are working with general configuration spaces. Second, on a general configuration space, there is no such thing as a straight line. Thus we are led to the following definition. On a configuration space $Q$, we define the work done by a force $F$ along a curve $c: I \rightarrow Q$ to be the quantity

$$W(F, c) = \int_I \langle F(t, c'(t)); c'(t) \rangle \, dt.$$  

(2.14)

One should be able to readily see that this properly generalises the classical notion of work (see Exercise E2.27).

Now with this notion of work, we have the following characterisation of a potential force. A curve $c: [a, b] \rightarrow Q$ is closed if $c(a) = c(b)$.

2.5.3 Proposition Let $F$ be a force on a configuration space $Q$. The following statements are equivalent:

(i) the work done by $F$ on any closed curve is zero;

(ii) $F$ is a potential force.

Proof The implication (ii) \(\Rightarrow\) (i) is readily made. Indeed, if $F$ is a potential force, then $F(t, x_0) = -dV(q)$ for some function $V$. We then have

$$W(F, c) = \int_a^b \langle F(t, c'(t)); c'(t) \rangle \, dt$$

$$= -\int_a^b \langle dV(c(t)); c'(t) \rangle \, dt$$

$$= -\int_a^b \frac{dV(c(t))}{dt} \, dt$$

$$= -(V(c(b)) - V(c(a))) = 0,$$

for every closed curve $c: [a, b] \rightarrow Q$.

The implication (i) \(\Rightarrow\) (ii) goes as follows. Suppose that $F$ is a potential force and let $q_0 \in Q$. For $q \in Q$ define $V(q) \in \mathbb{R}$ by

$$V(q) = -\int_0^1 \langle F(c(t)); c'(t) \rangle \, dt,$$

where $c: [0, 1] \rightarrow Q$ is any curve satisfying $c(0) = q_0$ and $c(1) = q$. We will first show that the function $V$ defined in this manner does not depend on the choice made for the curve $c$. Let $c_1, c_2: [0, 1] \rightarrow Q$ be two curves satisfying $c_1(0) = c_2(0) = q_0$ and $c_1(1) = c_2(1) = q$. Define a curve $\sigma: [0, 1] \rightarrow Q$ by

$$\sigma(t) = \begin{cases} c_1(2t), & t \in [0, \frac{1}{2}] \\ c_2(2-2t), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Note that $\sigma(0) = \sigma(1)$ so that

$$\int_0^1 \langle F(\sigma(t)); \sigma'(t) \rangle \, dt = 0.$$

However, we also have

$$\int_0^1 \langle F(\sigma(t)); \sigma'(t) \rangle \, dt = \int_0^{\frac{1}{2}} \langle F(\sigma(t)); \sigma'(t) \rangle \, dt + \int_{\frac{1}{2}}^1 \langle F(\sigma(t)); \sigma'(t) \rangle \, dt$$

$$= \int_0^1 \langle F(c_1(t)); c_1'(t) \rangle \, dt + \int_1^0 \langle F(c_2(t)); c_2'(t) \rangle \, dt,$$

from which we conclude that

$$\int_0^1 \langle F(c_1(t)); c_1'(t) \rangle \, dt = \int_0^1 \langle F(c_2(t)); c_2'(t) \rangle \, dt.$$

Thus our definition of $V$ is independent of curve, as desired.

Finally, let us show that $F = -dV$, thus making $F$ a potential force. Let $(U, \phi)$ be a coordinate chart around $q \in Q$ and suppose that $\phi(q) = 0$. Let $B_\epsilon(0)$ be the ball of radius $\epsilon$ centred at the origin in $\mathbb{R}^n$, and suppose that $B_\epsilon(0) \subset \phi(U)$. Let $v \in \mathbb{R}^n$ and let $x_1 = \frac{v}{\|v\|}$ be the point on $B_\epsilon(0)$ which lies in the direction of $v$. Denote $q_1 = \phi^{-1}(x_1)$. Let $c: [0, 1] \rightarrow Q$ be a curve with the properties

1. $c(0) = q_0$;
2. $c(\frac{1-t}{\|v\|}) = q_1$;
3. $\phi \circ c(t) = \frac{\phi(1-t)}{\|v\|} x_1$, $t \in [\frac{1-t}{\|v\|}, 1]$.

Now we define $\psi: [\frac{1-t}{\|v\|}, 1] \rightarrow \mathbb{R}$ by

$$\psi(t) = \int_{\frac{1-t}{\|v\|}}^t \langle F(\phi \circ c(s)); v \rangle \, ds,$$

where $F$ is the representation of $F$ is the coordinate chart $(U, \phi)$. Let $V_\phi: B_\epsilon(0) \rightarrow \mathbb{R}$ be defined by $V_\phi(x) = V \circ \phi^{-1}(x)$. Note that with $\psi$ defined as we have done, we have

$$V_\phi(v) = \psi(1-t), \quad t \in [0, \frac{1-t}{\|v\|}].$$
Therefore we have
\[ \left\{ dV_\phi(0); v \right\} = -\frac{d}{dt} \bigg|_{t=0} \psi(t). \]

By the definition of \( \psi \) this gives
\[ \left\{ dV_\phi(0); v \right\} = -\langle F(0); v \rangle. \]

If \( v_q \in T_q Q \) is the tangent vector whose coordinate representation in the chart \((U, \phi)\) is \( v \), then this means that
\[ \langle dV(q); v_q \rangle = -\langle F(q); v_q \rangle. \]

Since our above construction can be done for any \( v \in \mathbb{R}^n \) this shows that \( F(q) = -dV(q) \), as desired.

Of course, we have already seen a potential function, but let us explicitly identify the force associated with it.

2.5.4 Example (Example 2.1.5 cont’d) We consider the gravitational potential introduced in Example 2.4.3. In Cartesian coordinates we had \( V(x, y) = ma_y(y - y_0) \), so the potential force is simply
\[ F = -dV = -ma_y dy. \]

In polar coordinates we had \( V = ma_y (r \sin \theta - r_0 \sin \theta_0) \), so that the potential force in these coordinates is
\[ F = -dV = -ma_y (\sin \theta dr + r \cos \theta d\theta). \]

Were one to derive from “first principles” the formula for the potential force, one would have some work to do, but using the blind calculations, it is quite simple.

2.5.3 Dissipative forces A force \( F \) on a configuration space \( Q \) is a dissipative force if
\[ \langle F(t, v_q); v_q \rangle \leq 0 \]
for every \( (t, v_q) \in \mathbb{R} \times \mathbb{R}^n \). A dissipative force is strictly dissipative if strict inequality holds for each \( v_q \in T_q Q \). Dissipative forces “dissipate energy.” Although we have talked about kinetic energy and potential energy, we have not really talked about the total energy of a Lagrangian system. Certainly, for a simple mechanical system where we have talked about kinetic energy and potential energy, we have not really talked about the dissipation of energy. The dissipation of energy is a dissipative force, which is the whole point of the dissipative forces.

2.5.5 Lemma \( FL(t, v_q) \in T^* Q \).

Proof We need to show that the coordinate expression transforms in the correct way when we change coordinates. If \((q^1, \ldots, q^n)\) and \((\hat{q}^1, \ldots, \hat{q}^n)\) are coordinates in overlapping coordinate charts, then we have
\[ \frac{\partial L}{\partial \hat{q}^i} = \frac{\partial L}{\partial q^j} \frac{\partial q^j}{\partial \hat{q}^i} + \frac{\partial L}{\partial v^i}. \]

But \( q^j, j = 1, \ldots, n, \) is independent of \( \hat{v}^i, i = 1, \ldots, n \) (the \( q^j \)‘s are only functions of the \( \hat{q}^j \)’s, not of the \( \hat{v}^i \)’s). Also, we have
\[ \hat{v}^i = \frac{\partial \hat{q}^i}{\partial \hat{v}^i}, \]
so that
\[ \frac{\partial \hat{v}^i}{\partial \hat{v}^i} = \frac{\partial \hat{q}^i}{\partial \hat{v}^i}. \]

The result now follows from Lemma 2.2.11.
2.5.7 Proposition Let \( L \) be a time-independent Lagrangian on a configuration space \( Q \) and let \( F \) be a dissipative force. If a curve \( c: [a, b] \to Q \) satisfies the Lagrange-d’Alembert principle for the Lagrangian \( L \) and the force \( F \), then

\[
\frac{d}{dt} EL(t, c'(t)) \leq 0
\]

for every \( t \in [a, b] \).

Proof The proof is easily carried out in coordinates, using the fact that \( L \) is time-independent. Using the coordinate expression (2.15) for \( EL \) we have

\[
\frac{d}{dt} EL(t, c'(t)) = \frac{\partial EL}{\partial q^i} \dot{q}^i + \frac{\partial EL}{\partial \dot{q}^i} \ddot{q}^i
\]

which gives the coordinate independent expression,

\[
\frac{\partial^2 L}{\partial v^i \partial \dot{v}^i} \dot{v}^i + \frac{\partial^2 L}{\partial \dot{v}^i \partial \dot{v}^i} \ddot{q}^i + \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - \frac{\partial L}{\partial \ddot{q}^i} \ddot{q}^i - \frac{\partial L}{\partial \dot{v}^i} \dot{v}^i. \tag{2.16}
\]

Since the Euler-Lagrange equations hold, we may expand them fully to give

\[
\frac{\partial^2 L}{\partial v^i \partial \dot{v}^i} \dot{v}^i + \frac{\partial^2 L}{\partial \dot{v}^i \partial \dot{v}^i} \ddot{q}^i = F_i, \quad i = 1, \ldots, n. \tag{2.17}
\]

Comparing the expressions (2.16) and (2.17) gives the coordinate independent expression, true for general forces,

\[
\frac{d}{dt} EL(t, c'(t)) = \langle F(t, c'(t)); c'(t) \rangle.
\]

If \( F \) is a dissipative force, the result clearly follows.

This yields the following corollary of independent interest. It states that for an unforced Lagrangian system, energy is conserved.

2.5.8 Corollary If \( L \) is a time-independent Lagrangian on a configuration space \( Q \) then the function \( EL \) is constant along solutions of the Euler-Lagrange equations.

Let’s look at our simple example.

2.5.9 Example (Example 2.15 cont’d) We return to our example of a mass \( m \) moving in the plane \( Q = \mathbb{R}^2 \). We consider three examples of dissipative forces.

1. We take as an example of a dissipative force the force defined in Cartesian coordinates by

\[
F_1(t, x, y, v_x, v_y) = -\delta(v_x dx + v_y dy),
\]

where \( \delta > 0 \). That this is indeed a dissipative force follows since

\[
\langle F_1(t, x, y, v_x, v_y); (v_x, v_y) \rangle = -\delta(v_x^2 + v_y^2),
\]

and this quantity is negative except when \( v_x = v_y = 0 \). This kind of dissipative force is often called viscous friction, and is characterised by its being a linear function of velocity. It is often a good model when the contact is “lubricated.”

We can also represent this same force in polar coordinates. Trying to do this from “first principles” is a bit annoying. But just writing the above force in polar coordinates is quite simple: we compute

\[
v_x dx + v_y dy = (v_x \cos \theta - r v_y \sin \theta)(\cos \theta dr - r \sin \theta d\theta) +
(v_x \sin \theta + v_y \cos \theta)(\sin \theta dr + r \cos \theta d\theta)
\]

\[
= v_r dr + r^2 v_\theta d\theta.
\]

Therefore the viscous dissipation force in polar coordinates is

\[
F(t, r, \theta, v_r, v_\theta) = -\delta(v_r dr + r^2 v_\theta d\theta).
\]

2. Another example of a dissipative force is given by

\[
F_{K, \delta}(t, x, y, v_x, v_y) = \begin{cases} K(dx + dy), & v_x = v_y = 0 \\
-\delta(v_x dx + v_y dy), & \text{otherwise}, \end{cases}
\]

where both \( K \) and \( \delta \) are positive numbers. This kind of dissipative force is discontinuous, and is called sticking friction. It is commonly observed when there is no lubrication, and the contact between surfaces needs to be “broken” before motion can commence.

We may again compute the polar coordinate version of this force. Doing the calculations gives

\[
F_{K, \delta}(t, r, \theta, v_r, v_\theta) = \begin{cases} K((\cos \theta + \sin \theta) dr + (\cos \theta - \sin \theta) d\theta), & v_r = v_\theta = 0 \\
-\delta(v_r dr + r^2 v_\theta d\theta), & \text{otherwise}. \end{cases}
\]

Again, if one were to try to derive such a force in polar coordinates using first principles, it would not be a trivial task. But by understanding the force as a one-form, and so as a coordinate independent object, the calculations become systematic.

3. A final example of a dissipative force is what is termed rolling friction. This is the friction encountered when two surfaces are in rolling contact, and is intended to model the force required as one surface deflects as the other rolls across it. No matter, rolling friction is modelled by a force

\[
F_\alpha(t, x, y, v_x, v_y) = \begin{cases} 0, & v_x = v_y = 0 \\
-\alpha \frac{v_x}{\sqrt{v_x^2 + v_y^2}} dx - \alpha \frac{v_y}{\sqrt{v_x^2 + v_y^2}} dy, & \text{otherwise}. \end{cases}
\]

Note that this force is independent of the magnitude of velocity, and points in the opposite "direction" as velocity. As always, we may compute the rolling friction force in polar coordinate, and it is readily determined to be

\[
F_\alpha(t, r, \theta, v_r, v_\theta) = \begin{cases} 0, & v_r = v_\theta = 0 \\
-\alpha \frac{v_r}{\sqrt{v_r^2 + v_\theta^2}} dr - \alpha \frac{v_\theta}{\sqrt{v_r^2 + v_\theta^2}} d\theta, & \text{otherwise}. \end{cases}
\]

We shall leave to the reader the simple task of verifying that both sticking and rolling friction meet our criteria of being dissipative forces. □
2.5.4 Forces for simple mechanical systems When dealing with simple mechanical systems, because of their special structure, one can write forces in a slightly different manner, making them compatible with the writing of the equations using the affine connection formalism. The idea is quite simple, and is encapsulated in the following result, which gives a nice coordinate free way of expressing the forced equations for a simple mechanical system.

2.5.10 Proposition Let \((Q, g, V)\) be a simple mechanical system with Lagrangian \(L\), and let \(F\) be a force on \(Q\). A curve \(c: [a, b] \rightarrow Q\) satisfies the Lagrange-d’Alembert principle for \(F\) if and only if
\[
\dot{\nabla}_{t_0} c'(t) = -\nabla V(c(t)) + g^l(F(t, c'(t))).
\]

Proof This follows directly from Proposition 2.5.2 and the computations of Proposition 2.4.5.

As always, this is readily illustrated via our running example.

2.5.11 Example (Example 2.1.5 cont’d) Let us consider the particle of mass \(m\) of Example 2.4.1 with the potential function of Example 2.4.3 and the viscous friction force of Example 2.5.9. Therefore, in Cartesian coordinates, the force on the mass is
\[
F = -\delta (v_x dx + v_y dy).
\]

If we wished, we could consider the potential force as part of the total force, or we could leave it in the Lagrangian as the potential energy function; the equations are the same, of course. Just make sure you do not account for the potential force twice! The forced Euler-Lagrange equations, using the Lagrangian as the potential energy function; the equations are the same, of course. If we wished, we could consider the potential force as part of the total force, or we could leave it in the Lagrangian as the potential energy function; the equations are the same, of course. Just make sure you do not account for the potential force twice! The forced Euler-Lagrange equations, using the Lagrangian as the potential energy function; the equations are the same, of course.

We may compute the equations (2.18), with the aid of our computations in Example 2.4.6, to be
\[
\dot{r} - \dot{a}_0 \sin \theta = -\frac{\delta}{m} r,
\]
\[
\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} = -\frac{1}{m} \ddot{a}_0 \cos \theta - \frac{\delta}{m} \dot{\theta}.
\]

One readily sees that equations (2.21) and (2.22) are indeed equivalent.

2.6 Constraints in mechanics

The subject of constraints in mechanics seems to be one which lends itself readily to notational confusion. However, this need not be the case. With our notion of an abstract configuration space and its tangent bundle, it is quite easy to give a coherent definition of a constraint, and then write the equations of motion in the presence of a constraint. Thus you will find the treatment in this section brief compared to the often seen treatment, and devoid of the bizarre posturing one typically encounters.

2.6.1 Definitions For us, a constraint will be a specification of an affine subspace \(C_q\) of the tangent space \(T_q Q\) for each \(q \in Q\). Recall from Example 1.1.4 that as \(C_q\) is an affine subspace, there exists a subspace \(l(C_q)\) of \(T_q Q\) so that
\[
C_q = \{ \eta \cdot u_q \mid u_q \in l(C_q) \}
\]
for some vector \(\eta \in T_q Q\). We call \(l(C_q)\) the linear part of the constraint. A constraint is linear if \(C_q = l(C_q)\) for each \(q \in Q\). For simplicity, we will ask that \(\dim(l(C_q))\) be independent of \(q\) and that for each \(q \in Q\) there exists a neighbourhood \(U\) of \(q_0\) and linear independent vector fields \(X_1, \ldots, X_r\) on \(U\) so that
\[
l(C_q) = \text{span}_{U} \{ X_1(q), \ldots, X_r(q) \}, \quad q \in U,
\]
and a vector field \(\eta\) on \(Q\) so that \(C_q = \eta(q) + l(C_q)\). The integer \(r\) is the rank of the constraint, denoted \(\text{rank}(C)\). These assumptions are almost always (but not always) satisfied for physical systems. Note that it may not be possible to find vector fields \(X_1, \ldots, X_r\) defined on all of \(Q\) which have this property. We shall typically denote a constraint by \(C\) where, as expected,
\[
C = \bigcup \{ C_q \}_{q \in Q}.
\]

A curve \(c: I \rightarrow Q\) satisfies the constraint \(C\) if \(c'(t) \in C_{c(t)}\) for each \(t \in I\). Thus we see that what a constraint does is places restrictions on the allowable velocities.

An example here might do some good.

2.6.1 Example Let us introduce a new example into the mix. As this is the first time we have seen the example, let us take this opportunity to describe its configuration space and its Lagrangian.
The example is a disk rolling upright on a flat surface as shown in Figure 2.9. The configuration space for the disk is \( Q = \mathbb{R}^2 \times T^4 \), where \( \mathbb{R}^2 \) gives the point where the disk touches the plane and where the angles in \( T^4 \) describe the two angles of rotation, “roll” and “spin.” For coordinates on \( Q \), we shall take \((x, y, \theta, \phi)\) as shown in Figure 2.9.

We suppose the disk to have a uniform mass distribution with support in a vertical plane. This will place constraints on the velocities allowed to us, as we shall now see. In Figure 2.10, we view the disk from directly above. A little thought with this picture, and one realises that the condition that the disk roll without slipping may be expressed as the condition

\[
\dot{x} = r \cos \theta \dot{\phi}, \quad \dot{y} = r \sin \theta \dot{\phi},
\]

where \( r \) is the radius of the disk. This means that the points \((x, y, \theta, \phi, v_x, v_y, v_\phi, v_\theta)\) in \( TQ \) which are allowed by the constraints must satisfy

\[
v_x - r \cos \theta v_\phi = 0, \quad v_y - r \sin \theta v_\phi = 0.
\]

We now must put this into the form we have given for a constraint. That is, we must write down the set of allowable velocities at each configuration \((x, y, \theta, \phi)\). We shall do this by writing down vector fields on \( Q \) which span the linear part of the constraint distribution at each point. To do this, we note that a vector field \( X \) given by

\[
X = X_r \frac{\partial}{\partial x} + X_y \frac{\partial}{\partial y} + X_\theta \frac{\partial}{\partial \theta} + X_\phi \frac{\partial}{\partial \phi}
\]

will satisfy the constraints if and only if the vector \((X_r, X_y, X_\theta, X_\phi)\) lies in the kernel of the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & -r \cos \theta \\
0 & 1 & 0 & -r \sin \theta
\end{bmatrix}.
\]

One readily ascertains that two such vector fields are

\[
X_1 = r \cos \theta \frac{\partial}{\partial x} + r \sin \theta \frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}, \quad X_2 = \frac{\partial}{\partial \theta}
\]

We therefore have

\[
C_{(x, y, \theta, \phi)} = \text{span}_\mathbb{R} \{ X_1(x, y, \theta, \phi), X_2(x, y, \theta, \phi) \}.
\]

We note that this is therefore a linear constraint. \(\Box\)

### 2.6.2 Holonomic and nonholonomic constraints

We have already seen that a constraint is linear when \( C_q = \ell(C_q) \) for each \( q \in Q \). Linear constraints are by far the predominant type, although others certainly occur. Within the set of linear constraints, however, there is an important distinction which can be made. Let \( q_0 \in Q \). Denote by \( M_{q_0} \) the set of points \( q \in Q \) for which there exists a piecewise differentiable curve \( c : [0, 1] \to Q \), satisfying the constraints, with the property that \( c(0) = q_0 \) and \( c(1) = q \). Thus \( M_{q_0} \) is the set of points reachable from \( q_0 \) with curves which satisfy the constraint. The set of points \( M_{q_0} \) will typically be some smooth surface in \( Q \) running through the point \( q_0 \), and so \( M_{q_0} \) will have a well-defined dimension. We shall suppose that this dimension is independent of the point \( q_0 \), something which is frequently true. A linear constraint \( C \) is holonomic if the \( \dim(M_{q_0}) = \text{rank}(C) \). A constraint which is not holonomic is called nonholonomic. Thus, the idea with an holonomic constraint is that one can only access as many dimensions in configuration space as directions are allowed by the constraints. Perhaps a newcomer would expect that this would always be the case. But the fact of the matter is that many linear constraints are in fact nonholonomic. If this were not so, you would not be able to park your car. Indeed, your car can be thought of as having a configuration space of dimension 5 (at least for present purposes): (1) 3 dimensions for the position and orientation of the car (say \((x, y, \theta)\)); (2) 1 dimension for the steering wheel angle; (3) 1 dimension for the drive wheel angle i.e., what makes the car move forward. However, you have direct access to only 2 of the 5 directions, one via the steering angle, and the other via the drive wheel angle.
Nonetheless, by properly using these two directions, you can move your car anywhere you want!

Let us give a simple example of an holonomic constraint.

2.6.2 Example We work with \( Q = \mathbb{R}^2 \setminus \{0\} \). We exclude the origin, as we shall see, because we want the constraint we define to satisfy the property that \( \mathrm{rank}(C) \) be independent of \( q \). For \( (x, y) \in Q \) we define

\[
C_{(x,y)} = \text{span}_\mathbb{R}\left\{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right\}.
\]

Thus \( C \) is a linear constraint of rank 1. We shall show that it is holonomic by explicitly rendering the equation linear with constant coefficients:

\[
\begin{align*}
\dot{x} &= -a(t)y \\
\dot{y} &= a(t)x.
\end{align*}
\]

Thus we have rendered the equation linear with constant coefficients:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

The curve \( \tau \mapsto (x(\tau), y(\tau)) \) then satisfies

\[
\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = e^{A\tau} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{bmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.
\]

Thus the set of points reachable from \( (x_0, y_0) \) by curves which satisfy the constraint must lie on the circle of radius \( \sqrt{x_0^2 + y_0^2} \) with centre at the origin. This is illustrated in Figure 2.11. Notice that the subspace \( C_q \subset T_qQ \) is tangent to the set \( M_{q_0} \) of reachable points. This is generally the case.

Figure 2.11 An holonomic constraint

![Figure 2.11 An holonomic constraint](image)

Note that with an holonomic constraint, we may as well specify \( M_{q_0} \) since the tangent vectors in \( C_{q_0} \) will always simply be tangent to \( M_{q_0} \). Indeed, one often does simply specify the subset of accessible points \( M_{q_0} \) rather than specifying the vectors tangent to this set. In such cases, one directly restricts the configuration space to \( M_{q_0} \), and so in some sense, one may as well just take \( M_{q_0} \) to be the configuration space. The following example, following from Example 2.6.2, illustrates this.

2.6.3 Example (Example 2.6.2 cont’d) We suppose that we wish to model a classical pendulum. Thus the model we use is that of a point mass at the end of a massless rod of length \( \ell \) (see Figure 2.12). We shall declare that the configuration space for the system is \( Q = \mathbb{R}^2 \setminus \{0\} \), and then add the constraint that all configurations \( (x, y) \) of the system satisfy \( \sqrt{x^2 + y^2} = \ell \). Note, however, that the set of all points which satisfy the constraint is exactly the set \( M_{q_0} \) for the constraint \( C \) of Example 2.6.2, provided that one chooses \( q_0 = (x_0, y_0) \) to satisfy \( \sqrt{x_0^2 + y_0^2} = \ell \). Thus, in principle, there is not much difference between writing the constraint as \( C \) or by asking that all points satisfy the relation \( \sqrt{x^2 + y^2} = \ell \).

It is also now obvious that by taking \( Q = \mathbb{R}^2 \setminus \{0\} \), we have a configuration space that is simply too large for the problem. To fix this, we simply take our actual configuration space to be

\[
Q = \{(x, y) \in Q \mid \sqrt{x^2 + y^2} = \ell \}.
\]

The smaller configuration space is therefore \( S^1 \) in this example. As we shall see, it is of no consequence whether one chooses to define a holonomic constraint by specifying \( C \) or by specifying the smaller configuration space.

Let us now address the question of how one may determine whether a given linear constraint is holonomic or nonholonomic. By using the definition, or by following Example 2.6.2, we would have to construct all curves which start at a given point, and try to determine
the dimension of the set of points reached in this manner. Clearly this is not an attractive proposition. The following result, which we shall not prove, is a famous one, and relies on the Lie bracket which we defined in coordinates in (2.12). We refer to [van der Schaft and Maschke 1994] for a proof of this theorem.

2.6.4 Frobenius’s Theorem A linear constraint \( C \) is holonomic if and only if the vector field \([X, Y]\) takes values in \( C \) for every pair of vector fields \( X \) and \( Y \) taking their values in \( C \).

The idea, then, is simple. Although the statement of the result suggests that we need to take every pair of vector fields taking values in \( C \) and check their Lie bracket, in fact it suffices to choose a set of vector fields which form a basis, possibly only locally, for \( C \).

This is best illustrated with an example.

2.6.5 Example (Example 2.6.1 cont’d) We continue on with our example of the rolling disk.

As we have seen, the constraint in this example is linear, and has as basis the following two vector fields:

\[
X_1 = r \cos \theta \frac{\partial}{\partial x} + r \sin \theta \frac{\partial}{\partial y} + \frac{\partial}{\partial \phi}, \quad X_2 = \frac{\partial}{\partial \theta}.
\]

Theorem 2.6.4 tells us that to check whether the constraint is holonomic, we need only take the Lie bracket of the vector fields \( X_1 \) and \( X_2 \); if the resulting vector field satisfies the constraint, then the constraint is holonomic, if it does not then the constraint is nonholonomic.

The expression (2.12) for the Lie bracket tells us that the vector of components of the Lie bracket \([X_1, X_2]\) is the Jacobian of \( X_2 \) times \( X_1 \) minus the Jacobian of \( X_1 \) times \( X_2 \). Doing this gives

\[
D X_2 \cdot X_1 - D X_1 \cdot X_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Since the Lie bracket does not satisfy the constraint, the constraint is nonholonomic.

That the constraint should be nonholonomic in this example is not surprising, and let us explain why. Start the disk with the initial configuration \((0, 0, 0, 0)\) and with zero velocity. Were the constraint to be holonomic, then, from this configuration, we should only be able to reach a 2-dimensional subset of other configurations. However, it is possible to explicitly give a 3-dimensional subset of reachable configurations. There are two obvious directions which are reachable. The disk can “spin” and “roll.” Therefore, we can reach any configuration of the form \((r \theta, 0, \theta, \phi)\) where \( \theta \) and \( \phi \) are arbitrary. However, by combining a spin and a roll, one can see that it is possible to make the disk move in a circle of any radius. Therefore, it is possible to reach any configuration of the type \((0, y, x, 2\pi z)\) where \( y \) is arbitrary (the angle \(2\pi z\) is the angle through which the disk would have to roll in a semi-circular arc so that the point of contact would reach the point \((0, y))\). This then explicitly demonstrates a 3-dimensional subset of configurations which are reachable from \((0, 0, 0, 0)\), thereby precluding the constraint from being holonomic. \(\Box\)

A question which now arises is, “What does \( M_{\phi} \) look like when the constraint is not holonomic?” We know that when it is holonomic, the set of configurations available from a given configuration is restricted. Is the set of available configurations unrestricted when the constraint is nonholonomic? The answer is, “Maybe.” The fact is that the set of available configurations can in principle be described using constructions involving the Lie bracket,\(^3\) but we postpone this description until Section 4.5.1 where it will be given a control theoretic context. We will say here, however, that the dimension of the set of accessible configurations can be pretty arbitrary, ranging from a minimum possible value of \(\text{rank}(C)\) to a maximum possible value of \(\dim(Q)\). For the rolling disk example, it turns out that all points in \(Q\) are accessible from any point with curves which satisfy the constraint.

Let us finish off this section by giving a useful example of an holonomic constraint. This may seem like something of a contrived and silly example, but we shall see in Section 2.7 that it is actually quite useful.

2.6.6 Example We let \( Q = (\mathbb{R}^3)^6 \) be six copies of \( \mathbb{R}^3 \). Let \( r_1, r_2, \) and \( r_3 \) be positive real numbers. We shall construct an holonomic constraint by directly specifying the subset \( \tilde{Q} \) of points which we will consider. Given \((x_1, \ldots, x_6) \in Q\) define \( y_i = x_i - x_{i+3}, i = 1, 2, 3\). We shall say that a point \( q = (x_1, \ldots, x_6) \in \tilde{Q} \) is in the subset \( \tilde{Q} \) if it satisfies the conditions

1. \( \| y_i \| = 2r_i, i = 1, 2, 3, \)
2. the vectors \( y_1, y_2, y_3 \) are orthogonal,
3. the matrix

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
\]

has positive determinant, and
4. the three lines

\[\ell_i = \{ x + ty_i \mid t \in \mathbb{R} \}, \quad i = 1, 2, 3, \]

have a point of intersection.

The picture is that there is a point \( O \in \mathbb{R}^3 \) (the point of intersection \( \ell_1 \cap \ell_2 \cap \ell_3 \)) with the property that the points \( x_i \) and \( x_{i+3} \) lie a distance \( r_i \) from \( O \), and that the lines \( \ell_i, i = 1, 2, 3 \) are orthogonal. An attempt to illustrate this is Figure 2.13.

What we are trying to account for here is the configurations of a rigid body using the fact, demonstrated in Example 1.5.10–3, that a general rigid body with nondegenerate inertia tensor is dynamically equivalent to six appropriately placed points in \( \mathbb{R}^3 \). Therefore, the set \( \tilde{Q} \) we are claiming to be the configuration space of six point masses which are rigidly constrained so that they form a rigid body. We should therefore be able to provide an identification of \( Q \) with \( SE(3) \), since the latter is the set of configurations of a rigid body. Indeed, this is easily done as follows. Let \((x_1, \ldots, x_6) \in \tilde{Q} \) and define \( y_i = x_i - x_{i+3}, i = 1, 2, 3 \). Since the vectors \( y_1, y_2, y_3 \) are orthogonal and since the matrix

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
\]

has positive determinant, the vectors \( \frac{x_1}{\| x_1 \|}, \frac{x_2}{\| x_2 \|}, \frac{x_3}{\| x_3 \|} \) form a positively oriented orthonormal basis for \( \mathbb{R}^3 \). Therefore, there exists a unique matrix \( R \in SO(3) \) with the property that \( R e_i = \frac{x_i}{\| x_i \|}, i = 1, 2, 3 \), where \( e_i, e_2, e_3 \) is the standard basis for \( \mathbb{R}^3 \). Also, the point \( O = \ell_1 \cap \ell_2 \cap \ell_3 \in \mathbb{R}^3 \) is uniquely defined by our point in \( \tilde{Q} \). This therefore gives a unique

\(^3\)The result is often call “Chow’s theorem” [Chow 1939].
element $(\mathbf{R}, O)$ of $SE(3)$ which we can associated to each point in $\tilde{Q}$. One can also easily provide a unique point in $\tilde{Q}$ for each element of $SE(3)$, which establishes that indeed $\tilde{Q}$ is the configuration space for a rigid body.

### 2.6.3 The Euler-Lagrange equations in the presence of constraints

Up till now, our discussion of constraints has focused on how to write constraints and how to classify them. We have not told you how to actually give dynamical equations which describe the motion of a constrained system. It turns out that to do this, the correct thing to do is to add forces to the Euler-Lagrange equations which are “constraint forces.” These constraint forces have the property that they maintain the constraint. However, they should also have some other properties which ensure that the resulting equations do in fact agree with physical observations.

To provide the mechanical characterisation of constraint forces, for a constraint $C$, and a curve $c: [a, b] \to \tilde{Q}$ which satisfies the constraint, a virtual displacement along $c$ is an assignment of a tangent vector $v(t) \in T_{c(t)}Q$ for each $t \in [a, b]$ with the property that $v(t) \in \ell(C_{c(t)})$. We then define a constraint force along $c$ to be an assignment of a covector $\alpha(t) \in T^*_{c(t)}Q$ for each $t \in [a, b]$ which satisfies $(F(t); \dot{v}(t)) = 0$ for every virtual displacement $v$.

### 2.6.7 Remarks

1. A natural question to ask is why virtual displacements are not required to satisfy $v(t) \in C_{c(t)}$ rather than $v(t) \in \ell(C_{c(t)})$. Of course, the two competing definitions would agree when the constraints are linear, but when they are not, there is a genuine difference. Some authors choose to call $v$ a “possible displacement” when $v(t) \in C_{c(t)}$. The introduction of the additional terminology seems to have nothing to recommend it, however. I have not seen a reasonable description of why virtual displacements need to be defined the way we have defined them. Thus we must resort to the old mantra of, “Because it agrees with the physics.”

2. For linear constraints, where $C = \ell(C)$, constraint forces have the property that they do no work (in the sense of our definition (2.14)) on curves which satisfy the constraints. However, for constraints which are not linear, the constraint forces will generally do work on curves which satisfy the constraint.

We may now say what it means for a curve to be a solution for a Lagrangian system with a constraint.

### 2.6.8 Definition

Let $Q$ be a configuration space. A constrained Lagrangian system on $Q$ is a triple $(L, F, C)$ where $L$ is a Lagrangian on $Q$, $F$ is a force on $Q$, and $C$ is a constraint on $Q$. A curve $c: [a, b] \to \tilde{Q}$ is a solution to the constrained Lagrangian system $(L, F, C)$ if there exists a constraint force $\alpha$ so that together $c$ and $\alpha$ satisfy

(i) $c$ satisfies the constraint $C$ and

(ii) $\alpha$ satisfies the Lagrange-d’Alembert principle for the force $F + \alpha$.

Thus, in coordinates $(q^1, \ldots, q^n)$ for $Q$, the constrained equations are simply

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = F_i + \alpha_i, \quad i = 1, \ldots, n,$$

plus the requirement that $t \mapsto (q^1(t), \ldots, q^n(t))$ satisfies the constraints. Note that the constraint force $\alpha$ is an unknown in this equation. To make this more concrete, let us represent the constraint equations in a slightly different manner. For $q \in Q$, define a subspace of $T_q^*Q$ by

$$\text{ann}(\ell(C_q)) = \{ \alpha_q \in T_q^*Q \mid (\alpha_q; v_q) = 0 \text{ for every } v_q \in \ell(C_q) \}. $$

We also choose a vector field $\eta$ on $Q$ so that

$$C_q = \{ \eta(q) + u_q \mid u_q \in \ell(C_q) \}. $$

Thus $\eta(q)$ provides a vector to “shift” the subspace $\ell(C_q)$ to get the affine subspace $C_q$. The condition that a curve $c: I \to Q$ satisfy the constraint, i.e., that $c'(t) \in C_{c(t)}$, may be written as $c'(t) - \eta(c(t)) \in \ell(C_{c(t)})$ which is in turn the same as requiring that

$$(\alpha_q; c'(t) - \eta(c(t))) = 0, \quad \forall \alpha_q \in \text{ann}(\ell(C_q)).$$

To represent this in a set of coordinates we choose $n - r$ one-forms $\alpha^1, \ldots, \alpha^{n-r}$ on a coordinate neighbourhood so as to be a basis for $\text{ann}(\ell(C))$. A coordinate representation, $t \mapsto (\alpha^1(t), \ldots, \alpha^n(t))$, of a curve satisfies the constraint if and only if

$$\alpha^a q^a = \alpha^a, \quad a = 1, \ldots, n - r.$$

Thus we arrive at the following coordinate characterisation of the solutions for a constrained Lagrangian system.

### 2.6.9 Proposition

Let $(L, F, C)$ be a constrained Lagrangian system on $Q$. A curve $c: [a, b] \to Q$ is a solution to the constrained Lagrangian system $(L, F, C)$ if and only if for every chart $(U, \phi)$ for which $U$ intersects the image of $c$, the coordinate representation, $t \mapsto \phi(q(t))$, of $c$ satisfies the equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = F_i + \lambda \alpha^a_q, \quad i = 1, \ldots, n,$$

$$\alpha^a q^a = \alpha^a, \quad a = 1, \ldots, n - r,$$

where
(i) $\alpha^1, \ldots, \alpha^{n-r}$ are one-forms on $U$ forming a basis for $\text{ann}(\ell(C))$.
(ii) $\eta$ is a vector field satisfying (2.23), and
(iii) where $\lambda_1, \ldots, \lambda_n$ are functions of $t$ to be determined, and are called Lagrange multipliers.

**Proof** The only thing we have not already shown is that the unknown constraint force should be expressible as $\alpha = \lambda\alpha^n$. However, this is clear since by our definition of constraint forces, and since $\alpha^1, \ldots, \alpha^{n-r}$ form a basis for $\text{ann}(\ell(C))$.

2.6.10 Remark The $2n - r$ equations of Proposition 2.6.9 have both a differential and an algebraic component. Indeed, equations of this type are often called DAE’s, for “differential algebraic equations.” As such, methods for solution, including numerical methods, exist.

Let us see how this works out for our rolling disk example. It is really quite simple, as we shall see.

2.6.11 Example (Example 2.6.1 cont’d) We work with the coordinates $(x, y, \theta, \psi)$ as given in Example 2.6.1. Let us first determine one forms $\alpha^1, \alpha^2$ which form a basis for $\text{ann}(\ell(C))$. We have practically done this already. Recall that admissible velocities were specified to satisfy the relations

\[ v_x - r \cos \theta \dot{\psi} = 0, \quad v_y - r \sin \theta \dot{\psi} = 0. \]

From this, we can immediately “read off” the one-forms as

\[ \alpha^1 = dx - r \cos \theta d\phi, \quad \alpha^2 = dy - r \sin \theta d\phi. \]

Using the system Lagrangian $L = \frac{1}{2}(v_x^2 + v_y^2) + \frac{1}{2}I_\theta \dot{\theta}^2 + \frac{1}{2}J_\psi \dot{\psi}^2$, we then determine the equations of Proposition 2.6.9 to be

\[
\begin{align*}
mx &= \lambda_1 \\
m\ddot{y} &= \lambda_2 \\
I_\theta &= 0 \\
J_\psi &= -\lambda_1 r \cos \theta - \lambda_2 r \sin \theta \\
\dot{x} - r \cos \theta \dot{\psi} &= 0 \\
\dot{x} - r \sin \theta \dot{\psi} &= 0.
\end{align*}
\]

Let us not deal right now with the matter of solving these equations, but leave this until Chapter 3.

2.6.12 Remark In our above discussion we have presented a method for computing equations of motion for Lagrangian systems with constraints. There are other methods available for determining equations of motion. Some of these are now listed.

1. The **Poincaré equations**: This method is one which eliminates the need for Lagrange multipliers, by incorporating the equations of constraint into the equations of motion by introducing “quasi-velocities.” This method receives a quite coherent treatment in the recent book of Talman [2000].

2. The **Gibbs-Appell equations**: These equations use only “quasi-velocities,” but “quasi-accelerations.” In the original formulation of Appell [1900a, 1900b] and Gibbs [1879] (yes, the free energy guy), the treatment is for particles and rigid bodies. A generalisation to arbitrary Lagrangian systems is provided by Lewis [1996].

3. **Gauss’s principle of least constraint**: This method is of a variational nature, although Gauss [1829] was able to cast the problem so that the variational methods are finite dimensional. That this method bears a strong resemblance to the Gibbs-Appell equations is flushed out in a general setting by Lewis [1996].

4. The **“vakonomic method”**: Notice that the Lagrange-’d’Alembert principle, unlike the characterisation Theorem 2.3.2 of the Euler-Lagrange equations, is not given in the form of a variational problem. That is, the Lagrange-’d’Alembert principle does not produce equations which are necessary conditions for the minimisation of a functional on a class of curves. Since Definition 2.6.8 gives the definition of the solution of a constrained Lagrangian system in terms of the Lagrange-’d’Alembert principle in which the external force is a constraint force, it follows that one may expect that the solutions to a constrained system, even without external forces which are not constrained forces, are curves which minimise a functional on a class of curves. This is indeed the case, and is seen by some as a weakness of Definition 2.6.8. In an attempt to circumvent this difficulty, Kozlov [1983] gives a variational formulation of constrained motion, calling the resulting equations of “variational axiomatic kind,” leading to the name “vakonomic.” The problem is that these equations are genuinely different from the equations of Definition 2.6.8, as is shown in a simple example in [Lewis and Murray 1995]. There has been a heated debate over which of the methods, the vakonomic method or any of the methods equivalent to Definition 2.6.8, are correct [e.g., Kharlamov 1992, Kozlov 1992]. In Section 2.7 we shall show that the equations derived from Definition 2.6.8 can be derived from Newtonian mechanics. Therefore, if the vakonomic method is correct, it is so in violation of Newtonian mechanics. This constitutes, one should think, the death knell for the vakonomic method, at least as concerns its validity for modelling “normal” mechanical systems.

This is an incomplete list, and some people who read it will be offended in the extreme at the omission of their favourite method. However, with the exception of the “vakonomic method,” all methods are equivalent. Depending on the problem one is looking at, and what one wishes to do, one or the other of the available methods may be the most suitable.

2.6.4 Simple mechanical systems with constraints Let us briefly specialise our discussion of constraints to simple mechanical systems with Lagrangians of the form $L(v) = \frac{1}{2}g(v, v) - V(q)$. To present our main result, we need the notion of orthogonal projection. We let $(Q, g, V)$ be a simple mechanical system and we let $C$ be a constraint on $Q$. We let $\ell(C)^\perp$ be the $g$-orthogonal complement to $\ell(C)$. Thus for each $q \in Q$ we have a subspace of $T_qQ$ defined by

\[ \ell(C)^\perp = \{ v \in T_qQ \mid g(v, u) = 0 \text{ for all } u \in \ell(C) \}. \]

Now any vector $v \in TQ$ can be written as $v = v_1 + v_2$ with $v_1 \in \ell(C)$ and $v_2 \in \ell(C)^\perp$. We then define a map $P_C : TQ \to TQ$ which sends a vector $v$ to its component $v_2 \in \ell(C)^\perp$.

Let us see how to compute all of these objects in our rolling disk example.

2.6.13 Example (Example 2.6.1 cont’d) For the rolling disk we have $Q = \mathbb{R}^2 \times \mathbb{T}^2$ and we were using coordinates $(x, y, \theta, \phi)$. The Riemannian metric in these coordinates is

\[ g = m(dx \otimes dx + dy \otimes dy) + I d\theta \otimes d\theta + J d\phi \otimes d\phi, \]
and the one-forms which form a basis for $\text{ann}(\ell(C))$ are
\[ \alpha^1 = dx - r \cos \theta d\phi, \quad \alpha^2 = dy - r \sin \theta d\phi. \]

Referring to Exercise E2.35, we see that the vector fields $g^\ell(\alpha^1)$ and $g^\ell(\alpha^2)$ form a basis for $\ell(C)$. To compute the components of these vector fields, we may use matrix notation. The components of $g^\ell(\alpha^1)$ are then
\[ \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & J & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ -r \cos \theta \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{m} \\ 0 \\ \sin \phi \\ 0 \end{bmatrix}. \]
so that
\[ g^\ell(\alpha^1) = \frac{1}{m} \frac{\partial}{\partial x} - \frac{r}{J} \cos \theta \frac{\partial}{\partial \phi}. \]

We similarly compute
\[ g^\ell(\alpha^2) = \frac{1}{m} \frac{\partial}{\partial y} - \frac{r}{J} \sin \theta \frac{\partial}{\partial \phi}. \]

This then gives a basis for $\ell(C)$, so effectively describing it.

Let us now turn to the computation of the orthogonal projection $P_C$. This is actually a not entirely trivial thing to compute, but let us go through the steps. From Example 2.6.1 we have the vector fields
\[ X_1 = \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y} + \frac{\partial}{\partial \phi} \quad X_3 = \frac{\partial}{\partial \phi} \]
as a basis for $\ell(C)$. We also have the vector fields
\[ X_2 = \frac{1}{m} \frac{\partial}{\partial x} - \frac{r}{J} \cos \theta \frac{\partial}{\partial \phi} \quad X_4 = \frac{1}{m} \frac{\partial}{\partial y} - \frac{r}{J} \sin \theta \frac{\partial}{\partial \phi} \]
as a basis for $\ell(C)$. Therefore, any tangent vector $v$ can be written as
\[ v = v^1 X_1 + v^2 X_2 + v^3 X_3 + v^4 X_4 \]
for some suitable coefficients $(v^1, v^2, v^3, v^4)$. The orthogonal projection of $v$ onto $\ell(C)$ will then be simply
\[ P_C(v) = v^3 X_3 + v^4 X_4, \]
(2.25)

But it remains to do the computations. To determine $(v^1, v^2, v^3, v^4)$ we must use the change of basis formula. Thus we write
\[ \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{1}{r} & 0 & 0 & -\frac{1}{r} \cos \theta \\ 0 & \frac{1}{r} & 0 & -\frac{1}{r} \sin \theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \phi} \end{bmatrix}. \]

With the $4 \times 4$ matrix $P$ as defined above, if we write
\[ v = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_\phi \frac{\partial}{\partial \phi} = v^1 X_1 + v^2 X_2 + v^3 X_3 + v^4 X_4, \]
then we may determine $(v^1, v^2, v^3, v^4)$ in terms of $(v_x, v_y, v_\phi, v_\theta)$ using the change of basis formula:
\[ \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{bmatrix} = P^{-1} \begin{bmatrix} v_x \\ v_y \\ v_\phi \\ v_\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} mr(\cos \theta v_x + \sin \theta v_\phi) + Jv_\phi \nyms{2.6.8} \\ mr^2 \sin^2 \theta v_x - J mr J \cos \theta v_\phi \nyms{2.6.14} \\ -mr^2 \sin \theta \cos \theta \nyms{2.6.8} \\ mr^2 \cos \theta v_x - J mr J \sin \theta v_\phi \nyms{2.6.14} \end{bmatrix}. \]

Using (2.25) we may now compute the components of $P_C(v)$ relative to the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta})$ as
\[ \begin{bmatrix} v_x \\ 0 \\ 0 \nyms{2.6.14} \\ 0 \nyms{2.6.14} \end{bmatrix} \]

Thus the object with the brace under it is the matrix representation of $P_C$ at the point $q = (x, y, \theta, \phi)$ relative to the basis $\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta} \}$ for $T_q Q$. 

The example illustrates that unless we have to, we really do not want to participate in computing the projection $P_C$, and certainly we would wish to have at hand a computer to do the algebra. But, it is in principle computable.

The theorem of the previous example notwithstanding, let us see what our notation using $\ell(C)$ and $P_C$ gives us. The following result is useful because it gives us a way of writing the equations for constrained systems in a coordinate independent manner without resorting to a variational statement.

2.6.14 Proposition Let $(Q, g, V)$ be a simple mechanical control system with Lagrangian $L$, let $F$ be a force on $Q$, and let $C$ be a constraint on $Q$. Let $\eta$ be the unique vector field with the properties that $\eta(q) \in \ell(C)(q)$ and that
\[ C_q = \{ \eta(q) + v_q \mid v_q \in \ell(C)(q) \}. \]

Then a curve $c: [a, b] \rightarrow Q$ is a solution for the constrained system $(L, F, C)$ if and only for each $t \in [a, b]$ there exists $\lambda(t) \in \ell(C(c(t)))$ so that
\[ \frac{d}{dt} \eta(c(t)) = -\nabla V(c(t)) + g^L(F(t, c(t))) + \lambda(t) \]
\[ P_C(c'(t)) = \eta(c(t)). \]

Proof The result will follow from Propositions 2.5.10 and 2.6.9 provided we can show that the constraint force $a$ in Definition 2.6.8 has the property that $g^\ell(\alpha(c(t))) \in \ell(C)^\perp$. However, this follows since $g^\ell$: $T^*Q \rightarrow TQ$ maps $\text{ann}(\ell(C))$ exactly onto $\ell(C)^\perp$. This fact follows from Exercise E2.35. 
\[ \blacksquare \]
The essential content of the result is that for simple mechanical systems, when we apply $q^i$ to a constraint force, the resulting vector field is $g$-orthogonal to the linear part of the constraint distribution. This will be helpful for us when we look at establishing a correspondence between Newtonian and Lagrangian mechanics in Section 2.7.

Let us verify Proposition 2.6.14 for our rolling disk example.

2.6.15 Example (Example 2.6.1 cont’d) We have yet to compute the Christoffel symbols for the rolling disk, but this is easy since the coefficients of the Riemannian metric in the coordinates $(x, y, \theta, \phi)$ are constant, implying that all Christoffel symbols are zero. In Example 2.6.13 we determined that the vector fields

\[ X_3 = \frac{1}{m} \frac{\partial}{\partial x} - \frac{r}{g} \cos \theta \frac{\partial}{\partial y}, \quad X_4 = \frac{1}{m} \frac{\partial}{\partial y} - \frac{r}{g} \sin \theta \frac{\partial}{\partial \phi} \]

were a basis for $\ell(C)$. Therefore, the vector field $\lambda(t)$ of Proposition 2.6.14 must be of the form $\lambda(t) = \lambda^1(t)X_3(c(t)) + \lambda^4(t)X_4(c(t))$ for some functions $\lambda^1, \lambda^4: [a, b] \to \mathbb{R}$. Therefore, we may write the differential equation of Proposition 2.6.14 as

\[
\begin{align*}
\dot{x} &= \frac{1}{m} \lambda^1 \\
\dot{y} &= \frac{1}{m} \lambda^4 \\
\dot{\theta} &= 0 \\
\dot{\phi} &= -\frac{r}{g} \cos \theta \lambda^1 - \frac{r}{g} \sin \theta \lambda^4.
\end{align*}
\]

These equations are obviously equivalent to the first four of the equations (2.24). To these differential equations, we may append the constraint equations, which, in Proposition 2.6.14 are the equations $P_C(c'(t))$. Since we have computed $P_C$ in Example 2.6.13, we can in principle write down these equations, but let us not do this, as it is ugly, and at this stage pointless.

\[ \square \]

2.6.16 Remark The computations in this section with our disk example make one wonder about the actual utility of Proposition 2.6.14. While the equations in abstract form look nice, in practice to compute $P_C$ is burdensome. It is true that if one’s only objective is to write equations of motion, then computing $P_C$ is not recommended. However, it turns out that the formulation of the equations of motion in Proposition 2.6.14 has some interesting benefits, particularly in the study of control theory for these systems. It turns out that using the equations of Proposition 2.6.14 as background, one may put the equations of motion for a constrained simple mechanical control system into affine connection form, but using an affine connection which is not the Levi-Civita connection associated with the kinetic energy Riemannian metric. The control theory is explained in the paper [Lewis 2000b], while the affine connection background is explained in [Lewis 1998], motivated by earlier papers [e.g., Bloch and Crouch 1998, Synge 1928]. We shall deal with this briefly in Section 3.8.3. \[ \square \]

2.6.5 The Euler-Lagrange equations for holonomic constraints As a final element of our discussion of constrained systems, let us demonstrate that for holonomic constraints defined by their constraint surface $M_q$, one may genuinely restrict attention to $M_q$, forgetting that $M_q$ sits inside some larger configuration space.

2.6.17 Theorem Let $(L, F, C)$ be a constrained Lagrangian system and let $C$ be holonomic. Let $q_0 \in Q$ and define $Q = M_q$ to be the set of points accessible from $q_0$ by curves satisfying the constraint and let $TQ$ by the subset of $TQ$ consisting of vectors tangent to $Q$. Denote by $L$ the restriction of $L$ to $TQ$.

For a curve $c: [a, b] \to Q$ with the property that image$(c) \subset Q$, the following statements are equivalent:

(i) a curve $c: [a, b] \to Q$ is a solution to the constrained Lagrangian system $(L, C, F)$ with $c(a) \in Q$; 

(ii) there exists a force $\tilde{F}$ on $Q$ such that $c: [a, b] \to Q$ satisfies the Lagrange-d’Alembert principle for $\tilde{F}$ on $Q$.

Proof Let $c(t)$ be a point on image$(c)$ and let $(q^1, \ldots, q^n)$ be coordinates around $c(t)$. Let us choose the coordinates so that $(q^1, \ldots, q^r)$ are coordinates for $Q$ and so that $q^{r+1}, \ldots, q^{n+1} = 0$ on $Q$. (That this can be done is true, but is not obvious from anything we have said here.) The curve $c$ in these coordinates then looks like $t \mapsto (q^1(t), \ldots, q^n(t), 0, \ldots, 0)$. Since the vector fields $\frac{\partial}{\partial q^i}, i = 1, \ldots, r$ are tangent to $Q$, we can use $(q^1, q^2, v^1, v^2, \ldots, v^n)$ as coordinates for $TQ$. We then have

\[ L(t, q^1, \ldots, q^n, v^1, \ldots, v^n) = L(t, q^1, \ldots, q^n, 0, \ldots, 0, v^1, \ldots, v^n, 0, \ldots, 0). \]

We also note that $a^i = dq^{i-r+1}, \ldots, a^{n-r} = dq^n$ form a basis for ann$(\ell(C))$. We may now use the equations of Proposition 2.6.9 to represent in coordinates the solution of the constrained system:

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial q^a} \right) - \frac{\partial L}{\partial q^a} &= F_a, \quad a = 1, \ldots, r. \\
\frac{d}{dt} \left( \frac{\partial L}{\partial v^a} \right) - \frac{\partial L}{\partial v^a} &= F_b + \lambda_b, \quad b = n - r + 1, \ldots, n.
\end{align*}
\]

Since $q^{n-r+1}(t) = \cdots = q^n(t) = 0$ we have

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v^a} \right) = \frac{\partial L}{\partial v^a}(t, q^1(t), \ldots, q^n(t), 0, \ldots, 0, \dot{q}^1(t), \ldots, \dot{q}^n(t), 0, \ldots, 0), \quad b = 1, \ldots, n - r + 1, \ldots, n.
\]

For similar reasons we have

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial v^a} \right) &= \frac{\partial^2 L}{\partial v^a \partial v^b} + \frac{\partial^2 L}{\partial v^a \partial q^i} \dot{q}^i \\
&= \frac{\partial^2 L}{\partial v^a \partial q^i} + \frac{\partial^2 L}{\partial v^a \partial q^i} \dot{q}^i + \frac{\partial^2 L}{\partial v^a \partial q^i} \dot{q}^i, \quad a = 1, \ldots, r.
\end{align*}
\]

where the index $i$ runs from 1 to $n$, and the index $c$ runs from 1 to $r$, and where all partial derivatives are evaluated at $(t, q^1(t), \ldots, q^n(t), 0, \ldots, 0, \dot{q}^1(t), \ldots, \dot{q}^n(t), 0, \ldots, 0)$. One therefore sees that, in fact

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial v^a} \right) = \frac{d}{dt} \frac{\partial L}{\partial v^a}.
\]
Similarly,
\[ \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i} \]
when the left-hand side has the argument \((t, q^1(t), \ldots, q^n(t), 0, \ldots, 0, \dot{q}^1(t), \ldots, \dot{q}^n(t), 0, \ldots, 0)\).
Therefore, if we define a force on \(\tilde{Q}\) by
\[ \tilde{F}(t, q^1, \ldots, q^n, v^1, \ldots, v^n) = F(t, q^1, \ldots, q^n, 0, \ldots, 0, v^1, \ldots, v^n, 0, \ldots, 0), \]
the result follows.

This seems perhaps a bit intimidating, but it is very simple. The idea is that when the constraint is holonomic, then we may we well restrict to the subset \(\tilde{Q}\) specified by the constraint. Once we do this, we can also restrict the Lagrangian, and in doing so, we can then write the unconstrained Euler-Lagrange equations on \(\tilde{Q}\). The question then is, are the equations we get the same as the equations we get if we apply the definition of a constrained system? The above theorem answers this question in the affirmative.

This is easily illustrated with an example.

2.6.18 Example (Examples 2.1.5 and 2.6.2 cont’d) We look again at the pendulum example of Example 2.6.3. Thus we work with \(Q = \mathbb{R}^2 \setminus \{0\}\) and the holonomic constraint \(C\) which gives rise to the subset
\[ \tilde{Q} = \{(x, y) \in Q \mid x^2 + y^2 = \ell \} \]
as in Example 2.6.3. By the nature of \(\tilde{Q}\), obviously polar coordinates are the better coordinates for the job. Indeed, in polar coordinates we simply have
\[ \tilde{Q} = \{(r, \theta) \mid r = \ell \}. \]
Therefore, since \(r\) is fixed by restricting to \(\tilde{Q}\), we can simply use \(\theta\) as a coordinate. We then have
\[ T\tilde{Q} = \{(r, \theta, v_r, v_\theta) \mid r = \ell, v_\theta = 0\} \]
as the description of the tangent bundle \(T\tilde{Q}\) in polar coordinates. Thus we may use \((\theta, v_\theta)\) as coordinates for \(T\tilde{Q}\). The restricted kinetic energy will then be \(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\ell^2 \dot{\theta}^2\), simply the restriction of the kinetic energy to \(T\tilde{Q}\). Since the pendulum also needs potential energy, we also need to restrict the gravitational potential function. But from Example 2.4.3 we can easily see that the restriction of the potential function to \(\tilde{Q}\) is \(ma_\theta(\ell \sin \theta - r_0 \sin \theta_0)\). Thus the restricted Lagrangian, as a function of \((\theta, v_\theta)\), is
\[ L(\theta, v_\theta) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\ell^2 \dot{\theta}^2 - ma_\theta(\ell \sin \theta - r_0 \sin \theta_0). \]
The Euler-Lagrange equations for this Lagrangian are
\[ m\ddot{r} + ma_\theta \ell \cos \theta = 0. \tag{2.26} \]

Let us now compare this with what happens when we write the equations using Lagrange multipliers. First we need to find a basis for \(\text{ann}(\ell(C))\). But this is simple since in polar coordinates, the vector field \(\frac{\partial}{\partial \theta}\) is a basis for \(C\). Therefore \(dr\) will be a basis for \(\text{ann}(C)\). Thus we take \(a^1 = dr\). The equations of Proposition 2.6.9 then read
\[ m\dot{r} - r\dot{\theta}^2 = \lambda_1, \]
\[ m\dot{r}^2 + 2ma_\theta \dot{\theta} + ma_\theta \ell \cos \theta = 0 \]
\[ \dot{r} = 0. \]

From the first equation, using the fact that \(r(t) = \ell\), we read off \(\lambda_1 = -\ell \dot{\theta}^2\) which determines the constraint force. From the second equation, we directly see that it is equivalent to (2.26), using again the fact that \(r(t) = \ell\).

2.7 Newton’s equations and the Euler-Lagrange equations

By now, we have presented a rather general setting for Lagrangian mechanics, and although we have seen that things appear to be reasonable in a couple of examples, the discerning reader will question the very idea of Lagrangian mechanics as modelling the physical world, and will be crying out once more for Newton’s mechanics. This is hardly surprising since the contents of the chapter to this point have been a compression of about 250 years of intense work by many of the best mathematicians and mechanicians during that time. To attempt to quell some of these anxieties, in this section we shall prove that Lagrangian mechanics and Newtonian mechanics agree, at least when both can be applied. Clearly, Lagrangian mechanics is more general, deriving as it does from an abstract variational principle on a general configuration space. Thus we will provide a fairly general Newtonian setting for mechanics, and then show that the laws of motion of Newton are the same as the Euler-Lagrange equations.

2.7.1 Lagrangian mechanics for a single particle

We shall start with something very simple. We consider a single particle of mass \(m\) moving in \(\mathbb{R}^3\) and subject to a force and a constraint. Thus we have fixed a coordinate system \(\phi\) for some Galilean spacetime \(\mathcal{G} = (\mathcal{C}, V, g, \tau)\), and so the \(\mathbb{R}^3\) we are considering is the first component of the canonical Galilean spacetime \(\mathbb{R}^3 \times \mathbb{R}\). Let us first deal with the Newtonian mechanics for our particle. We shall consider the particle to have a force \(F\) acting on it. Thus, in the Newtonian setting, \(F\) is a map from \(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3\) to \(\mathbb{R}^3\) which sends a triple \((t, \mathbf{x}, \mathbf{v})\) of time, position, and velocity to a force. We also consider a constraint on the particle. Thus we constrain the velocities of the particle to satisfy equations of the form
\[ \mathbf{e}^a(x) \cdot \mathbf{v} = f^a(x), \quad a = 1, \ldots, 3 - r. \]
for maps \(\mathbf{e}^a : \mathbb{R}^3 \to \mathbb{R}^3\) and \(f^a : \mathbb{R}^3 \to \mathbb{R}, \quad a = 1, \ldots, 3 - r.\) Here \(r \in \{0, 1, 2, 3\}\), depending on how many constraints we place on the particle. Note that this form of writing the constraints follows along lines like the constraint equations of Proposition 2.6.9. This type of constraint will take into account any of the various ways in which the motion of a particle may be constrained; for example
1. the particle may be constrained to move on a surface in \(\mathbb{R}^3\),
2. the particle may be constrained to follow a prescribed path,
3. the particle may be constrained to slide on the surface of a rigid body, or
4. the particle may have its velocity constrained by some external source.

One may wish to consider generalisations where the maps \(\mathbf{e}^1, \ldots, \mathbf{e}^{3-r}\) are time-dependent. This, however, is easily done—one simply sticks a \(t\) in the argument of \(\mathbf{e}^a, a = 1, \ldots, 3 - r\), at each stage. We shall not worry about whether constraints are holonomic or nonholonomic—of the examples of constraints listed above, many are holonomic. As we have seen, we may think of an holonomic constraint as being nonholonomic. Therefore we will keep all constraints in the general nonholonomic form which allows holonomic constraints as a
special case. In Newtonian mechanics, experimental observation leads one to the conclusion that forces of constraint are always orthogonal to the direction of the linear part of the
constraint, with orthogonality being with respect to the standard inner product on \( \mathbb{R}^3 \). That is
to say, if you, for example, looked at any of the four constraint situations enumerated above,
you would measure that they behaved as if they were subjected to a force orthogonal to the
linear part of the constraint. Thus a constraint force will have the form
\[
R = \lambda_1 c^1(x) + \cdots + \lambda_{3-r} c^{3-r}(x),
\]
for some \( \lambda_1, \ldots, \lambda_{3-r} \). Newton’s equations are now written as
\[
\begin{align*}
\dot{m}\ddot{x} &= F(t, x, \dot{x}) + \lambda_1 c^1(x) + \cdots + \lambda_{3-r} c^{3-r}(x) \\
\dot{\alpha}^a(x, y, z)v^a &= \beta^a(x, y, z), \quad a = 1, \ldots, 3 - r,
\end{align*}
\]
where the functions of \( t, \lambda_1, \ldots, \lambda_{3-r} \), are to be determined.

This completes the Newtonian description of the dynamics of a single forced and con-
strained particle. For the Lagrangian description, let us fix upon using the canonical Carte-
sian coordinates \((x, y, z)\) for \( Q = \mathbb{R}^3 \) which give \((x, y, z, v_x, v_y, v_z)\) as the natural coordinates
for \( TQ \). The kinetic energy of the particle is then
\[
K = \frac{1}{2}m(v_y^2 + v_z^2 + v_z^2),
\]
giving the Riemannian metric
\[
g = m(dx \otimes dx + dy \otimes dy + dz \otimes dz)
\]
on \( Q \). We can subtract from this a potential function to get the classical Lagrangian, but
since the result is simply an external force, but one of a specific type, we shall omit the
potential function from the Lagrangian, and consider its appearance as part of the external
force. Indeed, now is a good time to describe a force in our Lagrangian setup. By our
definition, a force is to be a \( T^*Q \)-valued function with coefficients which may depend on
time, position, and velocity. Thus we write a force as
\[
F = F_x dx + F_y dy + F_z dz,
\]
where \( F \), as well as the coefficients \( F_x, F_y, \) and \( F_z \), have the argument \((t, x, y, z, v_x, v_y, v_z)\).
A constraint is then specifiable, as in Proposition 2.6.9, by a set of equations of the form
\[
\alpha^a_i(x, y, z)v^a = \beta^a(x, y, z), \quad a = 1, \ldots, 3 - r,
\]
for functions \( \alpha^a_i, a = 1, \ldots, 3 - r, i = 1, 2, 3, \) and \( \beta^a, a = 1, \ldots, 3 - r, \) on \( Q = \mathbb{R}^3 \). By
Proposition 2.6.9, the solution to the constrained system then satisfies the equations
\[
\begin{align*}
\dot{m}\ddot{x} &= F_x + \lambda_1 \alpha^a_1 \\
\dot{m}\ddot{y} &= F_y + \lambda_1 \alpha^a_2 \\
\dot{m}\ddot{z} &= F_z + \lambda_1 \alpha^a_3 \\
\alpha^a_i \dot{q}^a &= \beta^a, a = 1, \ldots, 3 - r.
\end{align*}
\]
(2.28)

However, if we define
\[
x = (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z) \\
F((t, \mathbf{v})) &= (F_x(t, x, y, z, v_x, v_y, v_z), F_y(t, x, y, z, v_x, v_y, v_z), F_z(t, x, y, z, v_x, v_y, v_z)) \\
c^a(x) &= (\alpha^a_1(x, y, z), \alpha^a_2(x, y, z), \alpha^a_3(x, y, z)),
\]
then clearly the equations (2.27) and (2.28) are equivalent; indeed they are identical!
Thus we have shown the following:

For a single particle moving in \( \mathbb{R}^3 \) subjected to a general constraint and a general
force, the equations of Newtonian mechanics and Lagrangian mechanics agree.

2.7 Lagrangian mechanics for multi-particle and multi-rigid body systems

Of course, the simplification of the previous section is too simple to be really all that useful.
Most mechanical systems one encounters are formed of various interconnections of particles
and rigid bodies. For example, a look through the exercises at the end of the chapter
will yield several mechanical systems which are built up of particles and rigid bodies. For
each of these, one could apply Newtonian methods, with appropriate constraints to model
the interconnectedness, and arrive at a set of governing equations. One could then choose
coordinates for the systems, possibly taking into account any holonomic constraints, and then
derive the Euler-Lagrange equations. The resulting set of equations would be equivalent to
those derived using Newtonian mechanics. In this section we shall prove that this is generally
the case. The first thing we need to do is simplify things by considering only particles and not
rigid bodies. We do this by resorting to Example 2.6.6 which shows that a rigid body with
nondegenerate inertia tensor can be regarded as six particles with an holonomic constraint
keeping them in a certain configuration. Using this fact, we can model any collection of
interconnected particles and rigid bodies by a collection of particles which are appropriately
constrained.

Let us proceed then by modelling a set of particles subject to arbitrary forces and
constraints in a Newtonian setting. We shall suppose that we have \( N \) particles with masses
\( m_1, \ldots, m_N \). To describe the system we use \((x_1, \ldots, x_N) \in (\mathbb{R}^3)^N\) to provide the
positions of the \( N \) masses. Each mass is subject to a force which may depend on time, and
on the position and velocity of the other \( N \) masses. Thus the force on mass \( i \) is written
\[
F_i(t, x_1, \ldots, x_N, v_1, \ldots, v_N).
\]
The collection of masses is also subject to constraints. Note that we do not constrain each mass independently since one can expect a constraint to relate two or more of the masses. Thus we have constraint equations of the form
\[
c_i^a(x_1, \ldots, x_N)v^a_1 + \cdots + c^a_i(x_1, \ldots, x_N)v^a_N = f^a_i(x_1, \ldots, x_N), \quad a = 1, \ldots, 3N - r, (2.29)
\]
for maps \( c^a_i : (\mathbb{R}^3)^N \rightarrow \mathbb{R}^3, i = 1, \ldots, N, a = 1, \ldots, 3N - r, \) and \( f^a_i : (\mathbb{R}^3)^N \rightarrow \mathbb{R}, a = 1, \ldots, 3N - r \). Here \( r \in \{0, \ldots, 3N - r\} \) depends on the number of constraints which we
apply to the system of particles. As with the single particle, we can readily make the
constraints time-dependent, but let us agree not to do this. The constraints place a force on
each mass. Again, we simply assert that the force of constraint on the \( i \)th mass is given by
a force of the form
\[
R_i = \lambda_1 c_i^a(x_1, \ldots, x_N) + \cdots + \lambda_{3N-r} c_i^{3N-r}(x_1, \ldots, x_N)
\]
for some \( \lambda_1, \ldots, \lambda_{3N-r} \). This is one of those facts that is empirically verified.

With the nature of the forces of constraint at hand, we can now immediately write down
Newton’s equations. These are
\[ \ddot{x}_1 = F_1(t, x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N) + \sum_{a=1}^{3N-r} \lambda_a c^a_1(x_1, \ldots, x_N) \]
\[ \vdots \]
\[ \ddot{x}_N = F_N(t, x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N) + \sum_{a=1}^{3N-r} \lambda_a c^a_N(x_1, \ldots, x_N) \]
\[ c^a_1(x_1, \ldots, x_N) \dot{x}_1 + \cdots + c^a_N(x_1, \ldots, x_N) \dot{x}_N = f^a(x_1, \ldots, x_N), \quad a = 1, \ldots, 3N-r. \] 
(2.30)

As usual, the functions of time, \( \lambda_1, \ldots, \lambda_{3N-r} \) are to be determined.

For the Lagrangian description, we take \( Q = (\mathbb{R}^3)^N \) and use coordinates \( (x_1, y_1, \ldots, x_N, y_N, z_N) \). We shall write the corresponding natural coordinates for \( TQ \) as \( (x_1, y_1, \ldots, x_N, y_N, z_N, v_1, \ldots, v_x, v_y, \ldots, v_{x_N}, v_{y_N}, v_{z_N}) \). The kinetic energy of the system is
\[ K = \frac{1}{2} m_1 (v^2_1 + v^2_y + v^2_z) + \cdots + \frac{1}{2} m_N (v^2_{x_N} + v^2_{y_N} + v^2_{z_N}). \]
This gives the Riemannian metric
\[ g = m_1 (dx_1 \otimes dx_1 + dy_1 \otimes dy_1 + dz_1 \otimes dz_1) + \cdots + m_N (dx_N \otimes dx_N + dy_N \otimes dy_N + dz_N \otimes dz_N). \]

A general force is written as
\[ F = F_{x_1} dx_1 + F_{y_1} dy_1 + F_{z_1} dz_1 + \cdots + F_{x_N} dx_N + F_{y_N} dy_N + F_{z_N} dz_N, \]
where the coefficients are functions of time, as well as functions of all the coordinates and their velocities. A general constraint, in the Lagrangian formalism, can be written using Proposition 2.6.9 as
\[ \alpha^a_1(q) v^i = \beta^a(q), \]
where \( q \) denotes a point in \( Q = (\mathbb{R}^3)^N \). One now directly applies Proposition 2.6.9 to get the equations of motion in the Lagrangian setting as
\[ m_1 \ddot{x}_1 = F_{x_1} + \sum_{a=1}^{3N-r} \lambda_1 \alpha^a_1, \quad m_1 \ddot{y}_1 = F_{y_1} + \sum_{a=1}^{3N-r} \lambda_1 \alpha^a_2, \quad m_1 \ddot{z}_1 = F_{z_1} + \sum_{a=1}^{3N-r} \lambda_1 \alpha^a_3 \]
\[ \vdots \]
\[ m_N \ddot{x}_N = F_{x_N} + \sum_{a=1}^{3N-r} \lambda_N \alpha^a_{3N-2}, \quad m_N \ddot{y}_N = F_{y_N} + \sum_{a=1}^{3N-r} \lambda_N \alpha^a_{3N-1}, \quad m_N \ddot{z}_N = F_{z_N} + \sum_{a=1}^{3N-r} \lambda_N \alpha^a_{3N} \]
\[ \alpha^a_1(q) v^i = \beta^a(q), \quad a = 1, \ldots, 3N-r. \] 
(2.31)

Now we define
\[ x_i = (x_i, y_i, z_i), \quad i = 1, \ldots, N, \quad v_i = (v_x, v_y, v_z), \quad i = 1, \ldots, N \]
\[ F_i = (F_{x_i}, F_{y_i}, F_{z_i}), \quad i = 1, \ldots, N \]
\[ c^a_i = (\alpha^a_{3N-2}, \alpha^a_{3N-1}, \alpha^a_N), \quad i = 1, \ldots, N, \quad a = 1, \ldots, 3N-r, \]
where we have omitted arguments for convenience. As with the single particle, one readily sees that the equations (2.30) and (2.31) are identical. Thus we have demonstrated the following:

For an arbitrary number of particles and rigid bodies moving in \( \mathbb{R}^3 \) subjected to a general constraint and a general force, the equations of Newtonian mechanics and Lagrangian mechanics agree.

2.7.1 Remark The above calculations are harder to write down than they are to understand. This is in contrast to the usual situation where much work is made of demonstrating the equivalence of the Newtonian and Lagrangian formalism. Part of the reason for this is that most authors deal explicitly with holonomic constraints and introduce a set of coordinates to parametrise the set \( M_q \). However, as we have demonstrated in Theorem 2.6.17, it matters not whether one restricts to the subset of \( Q \) defined by the holonomic constraints as far as the Lagrangian formalism is concerned. It therefore suffices to consider holonomic constraints as nonholonomic constraints, and if all one is after is a demonstration of the equivalence of the Newtonian and Lagrangian formalisms, then this is sufficient. \( \square \)

2.8 Euler’s equations and the Euler-Lagrange equations

In our above discussion of the equivalence of the Newtonian and the Lagrangian formulations for mechanics, we cheated somewhat when considering rigid bodies in that we modelled a rigid body as a collection of particles subject to constraints. While this is correct, it deprives us of any insight in understanding how Euler’s equations for rigid body motion are related to the Euler-Lagrange equations. Let us address this point by demonstrating explicitly the equivalence between the two sets of equations for a single rigid body in the absence of constraints and forces.

Thus in this section we consider a rigid body \((\mathbb{B}, \mu)\) moving in a Galilean spacetime \( \mathcal{G} = (\mathcal{E}, V, g, \tau) \) via the rigid motion \( \Sigma = (\Psi, \nu) \). We consider a centre of mass observer \( \mathcal{O} \in \text{Ob}(\Sigma) \). By choosing a coordinate system \( \phi \in \text{Coor}(\mathcal{O}) \), the equations which govern the rigid body are, by Proposition 1.6.12,
\[ \dot{R}(t) = R(t) \dot{\Omega}(t) \]
\[ \dot{r}(t) = R(t) \dot{V}(t) \]
\[ I_\Omega(\dot{\Omega}(t)) = (I_\Omega(\dot{\Omega}(t))) \times \Omega(t) \]
\[ \mu(\mathcal{B}) \dot{V}(t) = \mu(\mathcal{B}) V(t) \times \Omega(t), \]
where \( t \mapsto (R(t), r(t)) \) are coordinate representations of the rigid motion, and where \( \dot{\Omega}, \dot{V}, \) and \( I_\Omega \) are coordinate representations of the body angular velocity, the body linear velocity, and the inertia tensor. While these equations do indeed describe the motion of the rigid body, the problem is that it is not so easy to compare them to the Euler-Lagrange equations of the rigid body since the body velocities \( \dot{\Omega} \) and \( V \) are not expressible as derivatives of coordinates. Actually, you will recall that with our choice of a centre of mass observer and a coordinate system adapted to this observer, the linear body velocity \( \dot{V}(t) \) is in fact \( \dot{r}(t) \), and so is a legitimate Lagrangian velocity. However, the body angular velocity, being defined by \( \dot{\Omega}(t) = R(t) \dot{R}(t) \) is not actually a legitimate Lagrangian velocity. This is not entirely obvious, but is true (Exercise E2.37). This then raises the question about what is the best way to do Lagrangian mechanics for a rigid body. Let us now turn to this very issue.
2.8.1 Lagrangian mechanics for a rigid body

We retain the setup in the introduction to the section, and we try to fit this setup into our Lagrangian scheme. The configuration space for the rigid body is clearly \( SE(3) \). We shall represent a point in \( SE(3) \) by \((\mathbf{R}, \mathbf{r}) \in SO(3) \times \mathbb{R}^3 \). We need to assign a Lagrangian to the system. The rigid body has kinetic energy. To define the kinetic energy, we integrate the kinetic energy of each point in the rigid body via its mass distribution. The following result records the answer.

2.8.1 Lemma Let \((\mathcal{B}, \mu)\) be a rigid body in a Galilean spacetime \( \mathcal{G} = (\mathcal{E}, V, g, r) \) which undergoes a rigid motion \( \Sigma = (\Psi, \phi) \). Let \( \mathcal{E} \in Ob(\Sigma) \) and suppose that \( \phi \) is a coordinate system adapted to \( \mathcal{E} \). If \( t \mapsto (\mathbf{R}(t), \mathbf{r}(t)) \) are the coordinate representations of the rigid motion, \( t \mapsto \mathbf{v}(t) \) are the coordinate representations of the body angular and linear velocity, and \( \Omega_{\mathcal{E}}(t) \) is the coordinate representation of the inertia tensor about the centre of mass, then the kinetic energy of the rigid body along the rigid motion is

\[
t \mapsto \frac{1}{2} \mu(\mathcal{B}) \| \mathbf{v}(t) \|^2 + \frac{1}{2} g_{\text{can}}(\Omega_{\mathcal{E}}(t), \Omega_{\mathcal{E}}(t)).
\]

Proof Suppose that \( \mathcal{B} \subset \mathcal{E}(s_0) \). If \( T_\mathcal{B} \) is the motion associated with the rigid motion \( \Sigma \), then the kinetic energy is by definition

\[
\frac{1}{2} \int \| T_\mathcal{B}'(s, x) \| \, d\mu.
\]

However, with the notation we have introduced for our coordinate representation of the rigid motion, the coordinate form of \( T_\mathcal{B} \) is

\[
T_\mathcal{B}(t, x) = R(t)x + r(t),
\]

since the observer at each instant is at the origin in the coordinate system \( \phi \). Here \( x \) is a point in the rigid body \( \phi_{\mathcal{E}}(\mathcal{B}) \). Therefore the kinetic energy at time \( t \) is

\[
K(t) = \frac{1}{2} \int \| R(t)x + \dot{r}(t) \|^2 \, d\mu.
\]

We have \( R(t) = R(t)\Omega(t) + \dot{r}(t) = R(t)\mathbf{v}(t) \) using the definitions of body velocity. Thus we compute

\[
K(t) = \frac{1}{2} \int \| R(t)\mathbf{v}(t) \|^2 \, d\mu + \frac{1}{2} \int \| R(t)\Omega(t) \times x \|^2 \, d\mu + \int g_{\text{can}}(R(t)\mathbf{v}(t), R(t)\Omega(t) \times x) \, d\mu
\]

\[
= \frac{1}{2} \mu(\mathcal{B}) \| \mathbf{v}(t) \|^2 + \frac{1}{2} \int g_{\text{can}}(x \times (\Omega(t) \times x), \Omega(t)) \, d\mu
\]

\[
= \frac{1}{2} \mu(\mathcal{B}) \| \mathbf{v}(t) \|^2 + \frac{1}{2} g_{\text{can}}(\Omega_{\mathcal{E}}(t), \Omega_{\mathcal{E}}(t)).
\]

Here we have used Lemma 1.5.2(ii), the vector identity (1.19), and the definition of the inertia tensor.

As an immediate corollary to the lemma, we have the following description of the Riemannian metric describing the rigid body kinetic energy. Note that in the statement of the corollary, we are writing a point in \( T^\ast SO(3) \) as \((\mathbf{R}, \mathbf{A})\). Thus \( \mathbf{A} \) is a matrix representing a tangent vector in the tangent space \( T^\ast SO(3) \); one may wish to think of \( \mathbf{A} \) as being \( \dot{\mathbf{R}} \). We also write \( \mathbf{A}^\ast \) for the image of a \( 3 \times 3 \) skew-symmetric matrix \( \mathbf{A} \) under the inverse of the map \( 
\).

2.8.2 Corollary The Riemannian metric on \( Q = SE(3) \) describing the kinetic energy of Lemma 2.8.1 is given by

\[
g_{\text{FB}}(\mathbf{r}, \mathbf{A})(((\mathbf{v}_1, \mathbf{A}_1), (\mathbf{v}_2, \mathbf{A}_2))) = \mu(\mathcal{B}) g_{\text{can}}(\mathbf{v}_1, \mathbf{v}_2) + g_{\text{can}}(I((\mathbf{R}^{-1}\mathbf{A}_1)^\ast), (\mathbf{R}^{-1}\mathbf{A}_2)^\ast).
\]

Proof From the expression of Lemma 2.8.1 for the rigid body kinetic energy, the only thing we need to show is that any tangent vector \( \mathbf{A} \in T^\ast SO(3) \) can be written as \( \mathbf{R} \mathbf{A} \mathbf{R}^{-1} \) for some \( \mathbf{A} \in \mathfrak{so}(3) \). But this follows from the proof of Proposition 1.4.5 where we showed that if \( t \mapsto \mathbf{R}(t) \) is a curve in \( SO(3) \), then the tangent vector \( \mathbf{R}(t) \) in the tangent space \( T_{\mathbf{R}(t)}SO(3) \) always has the form \( \mathbf{R}(t)\Omega(t) \) for some \( \Omega(t) \in \mathfrak{so}(3) \). Since this is true for any curve \( t \mapsto \mathbf{R}(t) \), it must be true for any tangent vector at any point.

Lemma 2.8.1 provides us, then, with a description of the kinetic energy of a rigid body in terms of the body velocities. What one would like to do is use this expression as the Lagrangian for the rigid body since we are assuming there are no external forces or constraints. However, as we have seen, the body angular velocity is not a normal Lagrangian velocity, so we cannot use it as we use \( \mathbf{v} \) in the Euler-Lagrange equations. This leaves us with having to use coordinates for \( SE(3) \) to describe the motion of a rigid body in the Lagrangian setting. Since \( SE(3) = SO(3) \times \mathbb{R}^3 \) this amounts really to choose coordinates for \( SO(3) \). For this, one could use the Euler angles \((\alpha, \beta, \gamma)\) of Example 2.1.3–3. One would then have to express the velocities \((\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)\) in terms of the body angular velocity to get the kinetic energy expressed in a form where one could apply the Euler-Lagrange equations. To do this, one could differentiate the expression (2.3) with respect to \( t \) (suppose that \( (\alpha, \beta, \gamma) \) were functions of \( t \) and then use the relation \( \dot{\mathbf{R}}(t) = \mathbf{R}(t)\Omega(t) \). One would then have a Lagrangian given in terms of coordinates for \( TSO(3) \), and it would then be possible to compute the Euler-Lagrange equations. While this is possible, it is not very pleasant, and is not a very agreeable way to prove that the Euler equations are equivalent to the Euler-Lagrange equations.

In the next section we illustrate a rather more slick way to show the equivalence we are after.

2.8.2 A modified variational principle

The idea we give here is due to Marsden and Scheurle [1999] and is also explained in the book [Marsden and Ratiu 1999]. What we shall do is, rather than compute the Euler-Lagrange equations for a rigid body, we will employ the variational principle of Section 2.3 to deduce the equivalence of the Euler equations for a rigid body with the Euler-Lagrange equations. The result is the following important theorem.

2.8.3 Theorem On \( Q = SE(3) \) consider the Riemannian metric \( g_{\text{FB}} \) given by Corollary 2.8.2 and let \( L \) be the corresponding Lagrangian function on \( TSO(3) \). Define a function \( \ell \) on \( \mathbb{R}^3 \times \mathbb{R}^3 \) by

\[
\ell(v, \omega) = \frac{1}{2} \mu(\mathcal{B}) \| v \|^2 + \frac{1}{2} g_{\text{can}}(\mathbf{A}, \Omega_{\mathcal{E}}(t), \Omega_{\mathcal{E}}(t)).
\]

The following statements are equivalent:

(i) \( t \mapsto (\mathbf{R}(t), \mathbf{v}(t)) \) is a solution of the Euler-Lagrange equations with Lagrangian \( L \);
(ii) \( t \mapsto (\mathbf{R}^{-1}(t)\mathbf{r}(t), \mathbf{R}^{-1}(t)\mathbf{v}(t)) \) satisfies the Euler equations.

Furthermore, the above two statements are implied by the following:
Thus, we let

\[ \theta \]satisfies the conditions of their infinitesimal variations satisfy
\[
\frac{d}{ds}_{s=0} \omega(s, t) = \eta(t) + \omega(t) \times \eta(t) \]
\[
\frac{d}{ds}_{s=0} v(s, t) = u(t) + \omega(t) \times u(t) - \eta(t) \times v(t),
\]
where \( u \) and \( \eta \) vanish at the endpoints.

**Proof** Let us first show that (iii) is equivalent to the variational principle of Theorem 2.3.2. Thus, we let \( g : [a, b] \ni t \to (R(t), r(t)) \) be a curve on \( SE(3) \) and define a curve on \( \mathbb{R}^3 \times \mathbb{R}^3 \) by
\[
\xi(t) = (\Omega(t), V(t)) = (R^{-1}(t) \dot{r}(t), (R^{-1}(t) \dot{R}(t))^T).
\]

Let \( \sigma : (s, t) \to (r(s, t), R(s, t)) \) be a variation of a curve \( c \). We shall first show that the infinitesimal variation corresponding to \( \sigma \) gives rise to an infinitesimal variation of \( \xi \) which satisfies the conditions of (iii). Define \( \eta(t) = R^{-1}(t) \frac{d}{ds}_{s=0} R(s, t) \). Using the fact that
\[
\frac{d}{ds}_{s=0} R^{-1}(t) = -R^{-1}(t) \dot{R}(t) R^{-1}(t),
\]
we then have
\[
\dot{\eta}(t) = -R^{-1}(t) \dot{R}(t) R^{-1}(t) \frac{d}{ds}_{s=0} R(s, t) + R^{-1}(t) \frac{d}{ds}_{s=0} R(t).
\]
We then compute
\[
\frac{d}{ds}_{s=0} \Omega(s, t) = -R^{-1}(t) \frac{d}{ds}_{s=0} R(s, t) R^{-1}(t) \dot{R}(t) + R^{-1}(t) \frac{d}{ds}_{s=0} \dot{R}(t)\]
\[
= \dot{\eta} + (R^{-1}(t) \dot{R}(t)) (R^{-1}(t) \frac{d}{ds}_{s=0} R(s, t)) - \]
\[
(R^{-1}(t) \frac{d}{ds}_{s=0} R(s, t)) (R^{-1}(t) \dot{R}(t))\]
\[
= \dot{\eta} + \Omega(t) \dot{\eta}(t) - \dot{\eta} \dot{\Omega}(t)\]
\[
= \dot{\eta} + \Omega(t) \times \eta(t).
\]

In the last step we have used Exercise E1.11. We also need to show that variations of \( V(t) \) have the specified form. If we let \( u(t) = R^{-1}(t) \frac{d}{ds}_{s=0} r(s, t) \) we compute
\[
\dot{u}(t) = -R^{-1}(t) \dot{R}(t) R^{-1}(t) \frac{d}{ds}_{s=0} r(s, t) + R^{-1}(t) \frac{d}{ds}_{s=0} \dot{r}(t).
\]

Therefore
\[
\frac{d}{ds}_{s=0} V(s, t) = -R^{-1}(t) \frac{d}{ds}_{s=0} R(s, t) R^{-1}(t) \dot{r}(t) + R^{-1}(t) \frac{d}{ds}_{s=0} \dot{r}(t) = \dot{u}(t) + R^{-1}(t) \dot{R}(t) (R^{-1}(t) \frac{d}{ds}_{s=0} r(s, t)) - \]
\[
(R^{-1}(t) \frac{d}{ds}_{s=0} R(s, t)) (R^{-1}(t) \dot{R}(t)) = \Omega(t) u(t) - \eta(t) \dot{V}(t)\]
\[
= \dot{u}(t) + \Omega(t) \times u(t) - \eta(t) \times V(t).
\]

Now, since we have
\[
\int_a^b \ell(V(s, t), \Omega(s, t)) \, dt = \int_a^b L(R(s, t), \dot{R}(s, t)) \, dt,
\]
it follows that
\[
\frac{d}{ds}_{s=0} \int_a^b \ell(V(s, t), \Omega(s, t)) \, dt = 0
\]
will vanish for arbitrary variations if and only if
\[
\frac{d}{ds}_{s=0} \int_a^b L(R(s, t), \dot{R}(s, t)) \, dt
\]
vanishes for all variations of the form specified in part (iii) of the theorem.

By Theorem 2.3.2, the above calculations also show that (i) is implied by (iii).

Let us show that (iii) implies (ii). Suppose that we have a variation of \( t \to (V(t), \Omega(t)) \) satisfying the conditions of part (iii) of the theorem. We then compute
\[
\frac{d}{ds}_{s=0} \int_a^b \ell(V(s, t), \Omega(s, t)) \, dt = \int_a^b \left( \frac{d}{dt} \frac{d}{ds}_{s=0} \Omega(s, t) + \frac{d}{dt} \frac{d}{ds}_{s=0} V(s, t) \right) \, dt
\]
\[
= \int_a^b \left( -\frac{d}{dt} \frac{d}{ds}_{s=0} \Omega(t) + \frac{d}{dt} \frac{d}{ds}_{s=0} V(t) \right) \, dt + \frac{d}{ds}_{s=0} \left( \frac{d}{dt} \frac{d}{ds}_{s=0} \Omega(t) \times \eta(t) - \frac{d}{dt} \frac{d}{ds}_{s=0} V(t) \times \eta(t) \right) \, dt
\]
\[
= \int_a^b \left( -\frac{d}{dt} \frac{d}{ds}_{s=0} \Omega(t) + \frac{d}{dt} \frac{d}{ds}_{s=0} V(t) \times \eta(t) \right) \, dt + \frac{d}{ds}_{s=0} \left( \frac{d}{dt} \frac{d}{ds}_{s=0} \Omega(t) \times \eta(t) - \frac{d}{dt} \frac{d}{ds}_{s=0} V(t) \times \eta(t) \right) \, dt
\]
where all partial derivatives are evaluated at \( (\Omega(t), V(t)) \). Using the given expression for \( \ell \) we have
\[
\frac{d}{ds}_{s=0} \ell(\Omega(t), V(t)) = \mu(\mathbb{B}) V(t), \quad \frac{d}{ds}_{s=0} \ell(\Omega(t), V(t)) = \mathbb{L}(\Omega(t)).
\]

Therefore, using the fact that
\[
\frac{d}{ds}_{s=0} \int_a^b \ell(V(s, t), \Omega(s, t)) \, dt = 0
\]
for arbitrary variations gives
\[
\mathbb{L}(\Omega(t)) = (\mathbb{L}(\Omega(t))) \times \Omega(t) \mu(\mathbb{B}) V(t) = \mu(\mathbb{B}) V(t) \times \Omega(t).
\]
However, these are exactly the Euler equations, and thus this completes the proof. \( \square \)

This result provides a useful application of variational methods, although Theorem 2.8.3 strictly follows from the results of Section 2.7.
2.9 Hamilton's equations

The Hamiltonian formalism (named for Sir William Rowan Hamilton (1805–1865)) provides another setting in which to study the problems of mechanics. Some like to quibble over which is the “best” setting. We shall adopt the position that each provides advantages which can be exploited, and circumstances often dictate which, if any, approach is best. However, since the topic under discussion is Lagrangian mechanics, we will marginalise Hamiltonian mechanics only for this reason. There are many excellent texts which give a modern overview of Hamiltonian mechanics and dynamics [Abraham and Marsden 1978, Arne’l’d 1989, Libermann and Marle 1987, Tulman 2000].

Let L: R × TQ → R be a Lagrangian on Q. Recall that in Section 2.5.3 we had defined the Legendre transformation as a map FL: R × TQ → R × T*TQ which in a set of coordinates is given by

\((t, q^1, \ldots, q^n, v^1, \ldots, v^n) \mapsto \left(t, q^1, \ldots, q^n, \frac{\partial L}{\partial \dot{q}^1}, \ldots, \frac{\partial L}{\partial \dot{q}^n}\right).\)

We wish to think of the Legendre transformation as a change of coordinates, and compute the form of the Euler-Lagrange equations in these coordinates. This can only be done for certain Lagrangians. To wit, a Lagrangian L is hyperregular\(^6\) when FL is a diffeomorphism; that is, when FL is invertible and it and its inverse are smooth. The following result ensures that a large number of important Lagrangians are hyperregular.

2.9.1 Proposition If \((Q, \mathbb{R}, V)\) is a simple mechanical system with associated Lagrangian L, the Legendre transformation FL is hyperregular.

\[\frac{\partial F}{\partial q^i} \neq 0, \quad i = 1, \ldots, n.\]

Proof This is easily seen in coordinates where

\[FL(t, q^1, \ldots, q^n, v^1, \ldots, v^n) = (t, q^1, \ldots, q^n, g_{ij}v^i, \ldots, g_{ij}v^n).\]

From the coordinate expression (2.7) for the map \(g^i: TQ \to T^*Q\), we see that \(FL = g^i\). Since \(g^i\) is invertible with inverse \(g^i: T^*Q \to TQ\), the result follows. \(\blacksquare\)

Corresponding to a hyperregular Lagrangian \(L: \mathbb{R} \times TQ \to \mathbb{R}\) on Q define the corresponding Hamiltonian \(H_L: \mathbb{R} \times T^*Q \to \mathbb{R}\) by

\[H_L(t, \alpha_q) = EL(t, FL^{-1}(\alpha_q)) = \langle \alpha_q, FL^{-1}(\alpha_q) \rangle = L(t, FL^{-1}(\alpha_q)).\]

The following result gives the form of “Hamilton’s equations,” and how they are related to the Euler-Lagrange equations.

2.9.2 Theorem Let L be a hyperregular Lagrangian on a configuration space Q with \(H_L\) the corresponding Hamiltonian. Let c be a curve on Q and let (U, φ) be a coordinate chart for Q which intersects image(c). Denote natural coordinates for TQ in this coordinate chart by \((q^1, \ldots, q^n, v^1, \ldots, v^n)\) and denote natural coordinates for T*Q by \((q^1, \ldots, q^n, p_1, \ldots, p_n)\).

The following statements concerning c are equivalent:

(i) c satisfies the Euler-Lagrange equations

\[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, n;\]

(ii) the curve \(t \mapsto FL(c(t))\) satisfies Hamilton’s equations

\[
\dot{q}^i = \frac{\partial H_L}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_L}{\partial q^i}, \quad i = 1, \ldots, n.
\]

Proof In coordinates we have

\[H_L(t, q, p) = p_i v^i - L(t, q, v),\]

where we think of v as being expressed in terms of p using the inverse of the Legendre transformation. We thus compute

\[
dH_L = \frac{\partial H_L}{\partial q^i} dq^i + \frac{\partial H_L}{\partial p_i} dp_i + \frac{\partial H_L}{\partial t} dt = v^i dp_i + p_i \frac{\partial v^i}{\partial q^i} dq^i + p_i \frac{\partial v^i}{\partial p_i} dp_i - \frac{\partial L}{\partial \dot{q}^i} dq^i - \frac{\partial L}{\partial \dot{p}^i} dp_i - \frac{\partial L}{\partial t} dt.
\]

Now we use the fact that \(v^i\) is independent of \(q^j\) and the fact that \(p_j = \frac{d}{dt} q^j, j = 1, \ldots, n\), to simplify the above expression to

\[
\frac{\partial H_L}{\partial q^i} dq^i + \frac{\partial H_L}{\partial p_i} dp_i + \frac{\partial H_L}{\partial t} dt = v^i dp_i - \frac{\partial L}{\partial \dot{q}^i} dq^i - \frac{\partial L}{\partial \dot{p}^i} dp_i - \frac{\partial L}{\partial t} dt,
\]

where again v is to be thought of as a function of p via the inverse of the Legendre transformation. This gives us the equalities

\[\frac{\partial H_L}{\partial q^i} = -\frac{\partial L}{\partial \dot{q}^i}, \quad \frac{\partial H_L}{\partial p_i} = v^i.\]

Now suppose that the curve c satisfies the Euler-Lagrange equations. In this case, \(v^i = \dot{q}^i, i = 1, \ldots, n\), so the first n of Hamilton’s equations hold. Also, since \(p_i = \frac{d}{dt} q^i, i = 1, \ldots, n\), and since the Euler-Lagrange equations hold, we have

\[\frac{\partial H_L}{\partial q^i} = -\frac{\partial L}{\partial \dot{q}^i} = -\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i}\right) = -\dot{p}_i, \quad i = 1, \ldots, n.
\]

Thus the second n of Hamilton’s equations also hold.

Now suppose that Hamilton’s equations hold. One then easily sees that

\[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, n.
\]

This completes the proof. \(\blacksquare\)

The coordinates \(p_i, i = 1, \ldots, n\), are called conjugate momenta in the Hamiltonian formalism.

Let us see what Hamilton’s equations look like for our planar particle example.

2.9.3 Example (Example 2.1.5 cont’d) We first deal with Cartesian coordinates where the Lagrangian is, from Examples 2.4.1 and 2.4.3,

\[L(t, x, y, v_x, v_y) = \frac{1}{2} m (v_x^2 + v_y^2) - ma_y (y - y_0).
\]
The Legendre transformation is
\[ FL(t, x, y, v_x, v_y) = (t, x, y, mv_x, mv_y) \]
\[ \Rightarrow FL^{-1}(t, x, y, p_x, p_y) = (t, x, y, \frac{1}{m}p_x, \frac{1}{m}p_y). \]

From Proposition 2.5.6 we have
\[ EL(t, x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) + ma_y(y - y_0) \]
which then, by the definition of the Hamiltonian, gives
\[ H_L(t, x, y, p_x, p_y) = \frac{1}{2m}(p_x^2 + p_y^2) + ma_y(y - y_0) \]

We thus readily compute Hamilton’s equations in Cartesian coordinates to be
\[ \begin{align*}
\dot{x} &= \frac{\partial H_L}{\partial p_x} = \frac{p_x}{m} \\
\dot{y} &= \frac{\partial H_L}{\partial p_y} = \frac{p_y}{m} \\
\dot{p}_x &= -\frac{\partial H_L}{\partial x} = 0 \\
\dot{p}_y &= -\frac{\partial H_L}{\partial y} = -ma_y.
\end{align*} \]

One may readily verify that these equations have the same solution as their Lagrangian counterpart of Example 2.4.6.

Now let us do the same thing in polar coordinates where the Lagrangian is
\[ L(t, r, \theta, v_r, v_\theta) = \frac{1}{2}m(v_r^2 + r^2v_\theta^2) - ma_y(r \sin \theta - r_0 \sin \theta_0). \]

The Legendre transformation is then readily derived to be
\[ FL(t, r, \theta, v_r, v_\theta) = (t, r, \theta, mv_r, mr^2v_\theta) \]
\[ \Rightarrow FL^{-1}(t, r, \theta, p_r, p_\theta) = (t, r, \theta, \frac{1}{m}p_r, \frac{1}{mr^2}p_\theta). \]

The energy is easily seen to be given by
\[ EL(t, r, \theta, v_r, v_\theta) = \frac{1}{2}m(v_r^2 + r^2v_\theta^2) + ma_y(r \sin \theta - r_0 \sin \theta_0). \]

One then determines that in polar coordinates we have
\[ H_L(t, r, \theta, v_r, v_\theta) = \frac{1}{2m}(p_r^2 + \frac{1}{r^2}p_\theta^2) + ma_y(r \sin \theta - r_0 \sin \theta_0). \]

Hamilton’s equations in polar coordinates are then
\[ \begin{align*}
\dot{r} &= \frac{\partial H_L}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H_L}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{p}_r &= \frac{\partial H_L}{\partial r} = ma_y \sin \theta - \frac{p_\theta^2}{mr^3},
\end{align*} \]

Theorem 2.9.2 tells us that these equations must be the same as the Euler-Lagrange equations we derived in polar coordinates in Example 2.4.6. While the correspondence is easily established above in Cartesian coordinates, it is less transparent in polar coordinates. Indeed, although the moving between the Euler-Lagrange and the Hamiltonian equations are affected merely by a change of coordinates, there is something not completely transparent happening.

2.9.4 Remarks 1. For simple mechanical systems \((Q, g, V)\), as is the example we just worked out, it is possible to describe the Hamiltonian quite explicitly. For each \(q \in Q\) we may defined an inner product \(g^{-1}(q)\) on the cotangent space \(T^*Q\) by
\[ g^{-1}(q)(\alpha_q, \beta_q) = g(q^t(\alpha_q), q^t(\beta_q)). \]

One readily verifies that the matrix for this inner product with respect to a coordinate basis \(\{dq^1|_q, \ldots, dq^n|_q\}\) for \(T^*Q\) is simply the matrix with components \(g^{ij}, i, j = 1, \ldots, n\) (recall that the matrix with components \(g^{ij}\), \(i, j = 1, \ldots, n\), is defined to be the inverse of the matrix with components \(g_{ij}\), \(i, j = 1, \ldots, n\)). With this inner product on each cotangent space, the Hamiltonian corresponding to the Lagrangian \(L(v_q) = \frac{1}{2}g(v_q, v_q) - V(q)\) is given by \(H_L(v_q) = g^{-1}(v_q, v_q) + V(q)\).

An important distinction to make between the Lagrangian and the Hamiltonian formalisms comes up when describing the equations of motion associated with each. In the Hamiltonian setting, Proposition 2.9.5 provides a convenient equivalent expression for the Euler-Lagrange equations involving the Levi-Civita affine connection associated with the Riemannian metric \(g\). For Hamilton’s equations, there is no such convenient expression.

2. It is possible to include forces and constraints in an Hamiltonian setting, although we do not pursue this here [see Weber 1986].

3. As a final comment on the Hamiltonian approach, let us say that there is an enormous amount of literature available which expounds upon its many virtues. A good review is in the book [Libermann and Marle 1987]. An account of control theory for Hamiltonian control systems can be found in Chapter 12 of the book [Nijmeijer and van der Schaft 1990].

2.10 Conservation laws

In our discussion of rigid body dynamics in Section 1.6, we used the fact that spatial and angular momentum are conserved for a free rigid body to derive the Galilean Euler equations for rigid bodies. The idea of quantities being conserved by the motion of a system is an important one, and in the Lagrangian setting it has most basic formulation in Noether’s theorem. (Emmy Amalie Noether (1882–1935)) which we address in this section.

The formulation of Noether’s theorem requires the notion that a Lagrangian be “invariant” with respect to a vector field. Let us try to be clear about what that means. Let \(Q\) be a configuration space and let \(L : TQ \rightarrow R\) be a time-independent Lagrangian on \(Q\). For a vector field \(X\) on \(Q\), recall that the integral curve of \(X\) through \(q_0 \in Q\) is the unique curve \(t \mapsto c(t)\) with the property that \(c(0) = q_0\) and that \(c'(t) = X(c(t))\). (Also recall that \(c'(t)\) denotes the tangent vector field of \(c\). Thus \(c'(t) \in T_{c(t)}Q\) gives the “velocity” of the curve at time \(t\). The Lagrangian \(L\) is invariant under the vector field \(X\) if for each integral curve \(t \mapsto c(t)\), the function \(t \mapsto L(c(t))\) is independent of \(t\). The following result gives an easy way to determine when a Lagrangian is invariant under a vector field. The proof, although
readable with the tools we have developed thus far, perhaps relies on concepts just out of range of what we are doing.

2.10.1 Lemma Let $L: TQ \to \mathbb{R}$ be time-independent Lagrangian on a configuration space $Q$, and let $X$ be a vector field on $Q$. Let $(U, \phi)$ be a coordinate chart for $Q$ with coordinates $(q^1, \ldots, q^n)$. $L$ is invariant under $X$ if and only if
\[
\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial}{\partial q^i} \frac{\partial L}{\partial \dot{q}^i} \dot{v}^i = 0. \tag{2.32}
\]

Proof Note that the curve $c'(t)$ is the solution to the differential equation given in coordinates by
\[
\dot{q}^i = X^i, \quad \dot{v}^i = \frac{\partial X^i}{\partial q^j} v^j, \quad i = 1, \ldots, n. \tag{2.33}
\]
To verify this, one need only differentiate the first $n$ of these equations using the chain rule to get the second $n$ equations. Therefore, the equation (2.32) merely states that the directional derivative of $L$ in the direction $(X^1, \ldots, X^n, \frac{\partial X^i}{\partial q^j} v^j, \ldots, \frac{\partial X^n}{\partial q^n} v^n)$ is zero. But this means that $L$ is constant along solutions of the differential equations (2.33), which is exactly what is meant by $L$ being invariant under $X$, by definition. □

Let us see how this works out in an example.

2.10.2 Example (Example 2.15 cont’d) We take $Q = \mathbb{R}^2$ with the Lagrangian given in Cartesian coordinates by
\[
L(x, y, v_x, v_y) = \frac{1}{2} m (v_x^2 + v_y^2) - ma_y(y - y_0).
\]
Note here the Lagrangian is time-independent. Let us show that this Lagrangian is invariant under the vector field $X = \frac{\partial}{\partial t}$. One readily sees that the integral curve of $X$ through the point $(\tilde{x}, \tilde{y}) \in Q$ is given by $c: t \mapsto (\tilde{x} + t \tilde{y}, \tilde{y}) \in Q$. To arrive at this, one simply solves the initial value problem
\[
\dot{x} = 1, \quad \dot{y} = 0, \quad x(0) = \tilde{x}, \quad y(0) = \tilde{y}.
\]
Therefore, the corresponding curve $t \mapsto c'(t)$ is given by $t \mapsto (\tilde{x} + t \tilde{y}, 1, 0) \in TQ$. We then see that the function
\[
t \mapsto L(c'(t)) = \frac{1}{2} m - ma_y(y - y_0)
\]
is indeed independent of $t$, thus verifying that $L$ is indeed invariant under $X$, by directly employing the definition.

We may arrive at the same conclusion using Lemma 2.10.1. Indeed, this is merely a simple calculation. We note that the components of $X$ are $X^1 = 1$ and $X^2 = 0$. Therefore the terms in the expression (2.33) which involve the derivatives of the components of $X$ vanish. In this case, the expression (2.33) reduces to $\frac{\partial L}{\partial \dot{q}^1} X^1$ which is zero since $L$ is independent of $x$. In this way, and somewhat more straightforwardly, we again show that $L$ is invariant under $X$. □

The computations in the above example illustrate a commonly encountered situation. If $(U, \phi)$ is a coordinate chart, and a Lagrangian $L$ in this set of coordinates is independent of one of the coordinates, say $q^i$, then we say $q^i$ is a **cyclic coordinate** for $L$. As in the previous example, one may show that if $q^i$ is a cyclic coordinate, then $L$ is invariant under the vector field $X = \frac{\partial}{\partial q^i}$.

Now let us turn to investigating the implications of $L$ being invariant under some vector field. The following result shows that if $L$ admits a vector field $X$ under which it is invariant, then this implies the existence of a “constant of motion.”

2.10.3 Theorem (Noether’s Theorem) Let $L$ be a time-independent Lagrangian on $Q$ and let $X$ be a vector field on $Q$. Define a function $P_{L,X}: TQ \to \mathbb{R}$ by $P_{L,X}(v_q) = \langle FL(v_q), X(q) \rangle$. If $L$ is invariant under $X$, then the function $t \mapsto P_{L,X}(c'(t))$ is independent of $t$ for each solution $t \mapsto c(t)$ of the Euler-Lagrange equations.

Proof Fix a solution $t \mapsto c(t)$ of the Euler-Lagrange equations. For simplicity, suppose that $c(t)$ is defined for all $t \in \mathbb{R}$ and that all integral curves of $X$ can be infinitely extended. Define a map $\Phi: \mathbb{R} \times \mathbb{R} \to Q$ by asking that $\Phi(s, t) = F_X(s, c(t))$ where $F_X$ is the flow of $X$. Thus $\Phi(s, t)$ is the point on $Q$ where $c(t)$ gets mapped to after time $s$ under the flow of $X$.

1 Lemma For each $s \in \mathbb{R}$, the curve $t \mapsto \Phi(s, t)$ is a solution of the Euler-Lagrange equations.

Proof We use the variational characterisation of the Euler-Lagrange equations of Theorem 2.3.2. Since $c$ is a solution of the Euler-Lagrange equations, for $a < b$ we know that $c$ is an extremal of the functional
\[
\tilde{c} \mapsto \int_a^b L(c'(t)) \, dt
\]
over curves $\tilde{c} \in C^2(c(a), c(b), [a, b])$. Let us denote by $c_s$ the curve $t \mapsto \Phi(s, t)$. Since the Lagrangian is invariant under $X$, the function $L(c'_s(t))$ is independent of $s$. Therefore, $c_s$ is an extremal of the functional
\[
\tilde{c} \mapsto \int_a^b L(c'(t)) \, dt
\]
over curves $\tilde{c} \in C^2(c_s(a), c_s(b), [a, b])$, which means that $c_s$ is a solution of the Euler-Lagrange equations. ▼

For the remainder of the proof, we work in a coordinate chart for $Q$ with coordinates $(q^1, \ldots, q^n)$. In this case, $\Phi$ becomes a map from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^n$. By the lemma, for each fixed $s$ we have
\[
\frac{d}{ds} \frac{\partial L}{\partial \dot{q}^i} (\Phi(s, t), \dot{\Phi}(s, t)) - \frac{\partial L}{\partial q^i} (\Phi(s, t), \dot{\Phi}(s, t)) = 0, \quad i = 1, \ldots, n, \tag{2.34}
\]
where $\dot{\Phi}$ denotes the derivative of $\Phi$ with respect to $t$. Since $L(\Phi(s, t), \dot{\Phi}(s, t))$ is independent of $s$ we have
\[
\frac{d}{ds} \frac{\partial L}{\partial \dot{q}^i} (\Phi(s, t), \dot{\Phi}(s, t)) + \frac{\partial L}{\partial q^i} (\Phi(s, t), \dot{\Phi}(s, t)) = 0. \tag{2.35}
\]
Substituting (2.34) into (2.35) gives
\[
\frac{d}{ds} \frac{\partial L}{\partial \dot{q}^i} (\Phi(s, t), \dot{\Phi}(s, t)) + \frac{\partial L}{\partial q^i} (\Phi(s, t), \dot{\Phi}(s, t)) = 0.
\]
However, using the chain rule we write
\[
\frac{d\Phi'(s,t)}{ds} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\Phi}(s,t)} \right) + \frac{d\Phi'(s,t)}{ds} \frac{\partial L}{\partial \Phi(s,t)} = \frac{d}{dt} \left( \frac{\partial \Phi'(s,t)}{\partial \dot{\Phi}(s,t)} \frac{\partial L}{\partial \dot{\Phi}(s,t)} \right),
\]
which gives the result after evaluation at \( s = 0 \) since \( \frac{d}{ds} \Phi(s,t) = X(c(t)) \).

Noether’s theorem is often quite easy to apply. Let us do so for our simple example.

2.10.4 Example (Example 2.10.2 cont’d) We resume with the situation when the Lagrangian on \( Q = \mathbb{R}^2 \) given in Cartesian coordinates by
\[
L(x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2) - ma_y(y - y_0)
\]
is invariant under the vector field \( X = \frac{\partial}{\partial y} \). Theorem 2.10.3 tells us that the function on \( TQ \) given by
\[
\langle F_L(v_q); X(q) \rangle = \frac{\partial L}{\partial v_x} = mv_x
\]
is conserved. We recognise this simply as the linear momentum of the particle in the \( x \)-direction. Note that if we choose \( X = \frac{\partial}{\partial y} \) then the Lagrangian is no longer left invariant under \( X \) (check this!)

We could proceed to work through the above case in polar coordinates, but this is actually not very pleasant. Let us instead simplify the Lagrangian and look at another type of conservation law. We remove the potential energy from the Lagrangian so that in Cartesian coordinates we have
\[
L(x, y, v_x, v_y) = \frac{1}{2}m(v_x^2 + v_y^2).
\]
This Lagrangian will still be invariant under the vector field used above, but it is now invariant under other vector fields. Let us take the vector field \( X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \) and show that \( L \) is invariant under \( X \). Since \( L \) is independent of \( x \) and \( y \), and since \( X^1 \) is independent of \( x \) and \( X^2 \) is independent of \( y \), one readily computes that the expression (2.33) is given by
\[
\frac{\partial L}{\partial v_x} v_y + \frac{\partial L}{\partial v_y} v_x = m v_y (-1)v_x + m v_y (1)v_x = 0.
\]
This verifies that \( L \) is indeed invariant under \( X \). Noether’s theorem tells us that the corresponding conserved quantity is
\[
\frac{\partial L}{\partial v_x} (-y) + \frac{\partial L}{\partial v_y} (x) = m(xv_y - yv_x).
\]
If you are in the know, you recognise this as the angular momentum of the particle about the origin.

Let us now look at the situation in polar coordinates where
\[
L(r, \theta, v_r, v_\theta) = \frac{1}{2}m(v_r^2 + r^2 v_\theta^2).
\]
As we saw in Example 2.2.3, the vector field \( X \) is given by \( \frac{\partial}{\partial \theta} \). Also note that \( \theta \) is a cyclic coordinate! Therefore, had we been working in polar coordinates in the first place, the task of verifying that \( L \) is invariant under \( X \) would have followed immediately. In any case, the conserved quantity in polar coordinates is, by Theorem 2.10.3,
\[
\frac{\partial L}{\partial v_\theta} = m r^2 v_\theta.
\]
Perhaps this is more readily identified as the angular momentum of the particle about the origin. Indeed, the angular inertia of the particle is \( mr^2 \), and its angular velocity is \( v_\theta \), so the angular momentum is angular inertia times angular velocity.

Note that only when we used polar coordinates did we see that the situation was one where a cyclic coordinate was involved. Typically, one cannot expect to be in the cyclic situation, although it does come up often.

2.10.5 Remark If in some set of coordinates \( q^i \) is cyclic for the Lagrangian \( L \), then we see that the conserved quantity is \( \frac{\partial L}{\partial \dot{q}^i} \). But this is trivial, actually. If \( L \) is independent of \( q^i \), then the Euler-Lagrange equations directly give
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0,
\]
which is exactly the conservation law of Theorem 2.10.3. Thus one can view Noether’s theorem as a generalisation of cyclic coordinates.

\[\square\]
Exercises

E2.1 Suppose a mass is constrained to move in a three-dimensional Galilean sub-spacetime \( \mathcal{F} \) of a Galilean spacetime \( \mathcal{G} = (E, V, g, \tau) \).

(a) What is the configuration space for the system?
(b) How would one choose “Cartesian coordinates” for the system? Over which portions of the configuration space are these coordinates valid? What is the range of validity of these coordinates?
(c) How would one choose “polar coordinates” for the system? Over which portions of the configuration space are these coordinates valid? What is the range of validity of these coordinates?
(d) If the particle has mass \( m \), what is its kinetic energy in Cartesian coordinates?
(e) What is the kinetic energy of the particle in polar coordinates?

E2.2 Consider a particle constrained to move on the surface of a cylinder.

(a) What is the configuration space for the system?
(b) Find a set of coordinates for the configuration space, making sure you state exactly how they coordinatise the space, and their range of validity.
(c) Is it possible to find a single set of coordinates that are valid on the entire configuration space?
(d) Is it possible to find vector fields that form a basis each point of the configuration space?
(e) If the particle has mass \( m \), what is the kinetic energy for the system in the coordinates you specified?

E2.3 Let \( \mathcal{B} \) be a rigid body whose mass distribution has its support contained in a line (see Example 1.5.1–2).

(a) What is the configuration space for the system?
(b) Find a set of coordinates for the configuration space, making sure you state exactly how they coordinatise the space, and their range of validity.
(c) Is it possible to find a single set of coordinates that are valid on the entire configuration space?
(d) Is it possible to find two vector fields that are linearly independent at each point of the configuration space?
(e) Show that a single number \( I \) describes the inertia tensor of the system.
(f) With the coordinates you have chosen, provide an expression for the kinetic energy of the system.

E2.4 Consider a pendulum swinging atop a cart constrained to move in a line (Figure E2.1).

(a) What is the configuration space of the system?
(b) Define a coordinate system for the configuration space.

E2.5 Consider a pendulum attached to a radial arm by a universal joint. The base of the radial arm is constrained to move in a line. See Figure E2.2.

(a) What is the configuration space of the system?
(b) Define a coordinate system for the configuration space.

E2.6 Consider the two-axis gyroscope of Figure E2.3.

(a) Determine the configuration space for the system.

E2.7 Consider a point mass on the end of a massless leg, rotating on a rigid body fixed at a point in space (see Figure E2.4).

(a) Determine the configuration space for the system.
(b) Define a coordinate system for the configuration space.

E2.8 Consider the planar rigid body depicted in Figure E2.5.

(a) Determine the configuration space for the system.
(b) Define a coordinate system for the configuration space.

E2.9 Denote points on \( S^1 \subset \mathbb{R}^2 \) by \((\cos \theta, \sin \theta)\) where \( \theta \) takes values in \([0, 2\pi)\). Let \( f: S^1 \rightarrow \mathbb{R} \) be the function defined by \( f(\cos \theta, \sin \theta) = \theta \). Is \( f \) a continuous function on \( S^1 \)?

E2.10 Consider the map \( c: \mathbb{R} \rightarrow S^1 \) defined by \( c(t) = (\cos t, \sin t) \). Is \( c \) continuous? Differentiable?

E2.11 Define a vector field, any vector field, on \( S^1 \).

E2.12 Let \( Q = \mathbb{R} \) with \((x)\) the coordinate in the standard coordinate chart. Define a vector field on \( Q \) by \( X = x^2 \frac{\partial}{\partial x} \).

(a) Determine the flow of \( X \).
Exercises for Chapter 2

Now consider another coordinate chart \((V, \psi)\) for \(Q\) defined \(V = Q\) and \(\psi(x) = e^x\).

(b) Express \(X\) in this new coordinate chart.

(c) Determine the flow of \(X\) in this new coordinate chart.

E2.13 Is it possible to define two vector fields \(X_1\) and \(X_2\) on \(\mathbb{S}^2\) with the property that \(\{X_1(q), X_2(q)\}\) are linearly independent for every \(q \in Q\)?

E2.14 Let \(Q = \mathbb{R}^2\) with \((x, y)\) the coordinates in the standard coordinate chart. Define a vector field on \(Q\) by \(X = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}\).

(a) Determine the flow of \(X\).

(b) Determine the components of \(X\) in the polar coordinate chart.

(c) Determine the flow of \(X\) in the polar coordinate chart.

E2.15 Let \(Q\) be a configuration space with \((U, \phi)\) a coordinate chart. For vector fields \(X\) and \(Y\) with components \((X^1, \ldots, X^n)\) and \((Y^1, \ldots, Y^n)\) in the given chart, we propose the following two expressions,

\[
\begin{align*}
\frac{\partial Y^i}{\partial q^j} X^j + \frac{\partial X^i}{\partial q^j} Y^j, & \quad i = 1, \ldots, n, \\
\frac{\partial Y^i}{\partial q^j} X^j - \frac{\partial X^i}{\partial q^j} Y^j, & \quad i = 1, \ldots, n,
\end{align*}
\]

as being the components of a vector field on \(Q\). Which, if either, are actually the components of a vector field?

E2.16 When we allow arbitrary coordinate charts, the notion of a linear ordinary differential equation loses its distinctness. Let \(Q = \mathbb{R}^2\) and consider on \(Q\) with the standard coordinate system \((U_1, \phi_1)\) defined in the usual manner by \(U_1 = \mathbb{R}^2\) and \(\phi_1(x, y) = (x, y)\). In these coordinates define a vector field

\[
X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial x}.
\]

(a) Determine the flow of \(X\).

Now consider the coordinate chart \((U_2, \phi_2)\) where \(U_2 = \mathbb{R}^2\) and \(\phi_2(x, y) = ()\).

(b) Write the vector field \(X\) in this coordinate chart.

(c) Obtain the flow in these coordinates. Do you think you could have solved the differential equations were you given them without knowing how they came about?

E2.17 Let \(Q\) be a configuration space with \(g\) a Riemannian metric on \(Q\). Show that for a coordinate chart \((U, \phi)\) with coordinates \((q^1, \ldots, q^n)\) and for a fixed \(q \in U\), the linear map \(g^\flat(q) : T_qQ \to T_qQ\) has components \(g_{ij}\), \(i, j = 1, \ldots, n\), with respect to the bases \(\{\frac{\partial}{\partial q^i}\} \) and \(\{dq^1|_q, \ldots, dq^n|_q\}\) for \(T_qQ\) and \(T_qQ\), respectively. Also show that the components of \(g^\flat(q) : T_qQ \to T_qQ\) with respect to these same bases are \(g^{ij}\), \(i, j = 1, \ldots, n\).

E2.18 Let \(c \in C^2([a, b], q_0, q_0)\) and let \([t_1, t_2] \subset [a, b]\). Let \(\dot{c} \in C^2([t_1, t_2], c(t_1), c(t_2))\) and define a curve \(c_1 : [a, b] \to Q\) by

\[
c_1(t) = \begin{cases}
    c(t), & t \in [a, t_1] \\
    \dot{c}_1(t), & t \in [t_1, t_2] \\
    c(t), & t \in [t_2, b].
\end{cases}
\]
Exercises for Chapter 2

Show that for any \( \epsilon > 0 \) there exists a curve \( \tilde{c}_1 \in C^2([a,b],q_0) \) so that \( |J_L(\tilde{c}_1) - J_L(c_1)| \leq \epsilon \).

**Hint:** You may find it helpful to use a function like

\[
    f(x) = \begin{cases} 
    \exp(-1/(1-x^2)), & -1 < x < 1 \\
    0, & \text{otherwise.} 
    \end{cases}
\]

E2.19 For the system of Exercise E2.4 do the following.

(a) Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
(b) Determine the Christoffel symbols for the Riemannian metric.
(c) Determine the potential function.
(d) Using this data, write the Euler-Lagrange equations for the system.

E2.20 For the system of Exercise E2.5 do the following.

(a) Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
(b) Determine the Christoffel symbols for the Riemannian metric.
(c) Determine the potential function.
(d) Using this data, write the Euler-Lagrange equations for the system.

E2.21 For the system of Exercise E2.6 do the following.

(a) Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
(b) Determine the Christoffel symbols for the Riemannian metric.
(c) Determine the potential function.
(d) Using this data, write the Euler-Lagrange equations for the system.

E2.22 For the system of Exercise E2.7 do the following.

(a) Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
(b) Determine the Christoffel symbols for the Riemannian metric.
(c) Determine the potential function.
(d) Using this data, write the Euler-Lagrange equations for the system.

E2.23 For the system of Exercise E2.8 do the following.

(a) Determine the kinetic energy Riemannian metric for the system in your set of coordinates.
(b) Determine the Christoffel symbols for the Riemannian metric.
(c) Determine the potential function.
(d) Using this data, write the Euler-Lagrange equations for the system.

E2.24 Exercise on how the Newtonian potential gives the linear approximation.

E2.25 Let \( Q = \mathbb{R} \) and let \( F : \mathbb{R} \times TQ \to \mathbb{T}^*Q \) be a force which is independent of time and velocity. Show that \( F = 0 \) is a potential force. Show that this is not true when \( Q = \mathbb{R}^2 \).

E2.26 Let \( \nabla \) be an affine connection on a configuration space \( Q \) and let \( (U_1, \phi_1) \) and \( (U_2, \phi_2) \) be overlapping coordinate charts with coordinates \( (q^1, \ldots, q^n) \) and \( (\tilde{q}^1, \ldots, \tilde{q}^n) \). If \( \Gamma_{jk}^i \), \( i, j, k = 1, \ldots, n \) are the Christoffel symbols for \( \nabla \) in the coordinate chart \( (U_1, \phi_1) \) and \( \tilde{\Gamma}_{jk}^i \), \( i, j, k = 1, \ldots, n \) are the Christoffel symbols for \( \tilde{\nabla} \) in the coordinate chart \( (U_2, \phi_2) \), then

\[
    \tilde{\Gamma}_{jk}^i = \frac{\partial \tilde{q}^s}{\partial q^j} \frac{\partial \tilde{q}^r}{\partial q^k} \Gamma_{rs}^i + \frac{\partial \tilde{q}^s}{\partial q^j} \frac{\partial \tilde{q}^r}{\partial q^k} \frac{\partial q^i}{\partial q^s}.
\]

Show that as a result of this, the geodesic equations

\[
    \ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = 0, \quad i = 1, \ldots, n,
\]

are independent of coordinates. What can you say about the coordinate independence of each of the separate terms \( \ddot{q}^i \) and \( \Gamma_{jk}^i \dot{q}^j \dot{q}^k \) in the geodesic equation?

E2.27 Argue that the quantity defined in equation (2.14) generalises the usual freshman notion of “work.”

E2.28 Let \( Q = \mathbb{R}^3 \setminus \{ (0,0) \} \) and define a force on \( Q \) by

\[
    F(t, x, y, v_x, v_y) = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.
\]

(a) Is \( F \) a potential force? Why or why not?
(b) If \( Q = \{ (x, y) \in Q \mid x > 0 \} \), is \( F \) a potential force on \( Q \)? Why or why not?

E2.29 This is a continuation of Exercise E2.4. Suppose that a torque is applied to the base of the pendulum. What is this force, written as a one-form on \( Q \)?

E2.30 This is a continuation of Exercise E2.5. Suppose that two forces are applied to the system, one a torque at the base of the arm to rotate the arm, and the other a linear force moving the base down the track. What are these forces, written as a one-forms on \( Q \)?

E2.31 This is a continuation of Exercise E2.6. Suppose that a torque is applied which rotates the gyro in its frame. What is this force, written as a one-form on \( Q \)?

E2.32 This is a continuation of Exercise E2.7. Suppose that we actuate the system with two inputs as follows. We may apply a linear force to the mass, so extending its distance from the pivot, and we have a motor atop the rigid body which we can use to actuate the angle of the arm relative to the body. What are these forces, written as one-forms on \( Q \)?

E2.33 This is a continuation of Exercise E2.8. Suppose a force \( F \) is applied to the body at a point a distance \( h \) form the centre of mass. Write this force as a one-form on \( Q \).

E2.34 Show using Frobenius’ theorem that if \( C \) is a linear constraint with the property that rank(\( C \)) = 1, then \( C \) is holonomic.

E2.35 Let \( C \) be a constraint on \( Q \) and let the one-forms \( \alpha^1, \ldots, \alpha^{\text{rank}(\ell(C))} \) be a basis for \( \text{ann}(\ell(C)) \). Show that the vector fields \( g^1(\alpha^1), \ldots, g^1(\alpha^{\text{rank}(\ell(C))}) \) form a basis for \( \ell(\mathbb{C}) \).

E2.36 Consider a ball rolling on a flat horizontal table rotating with uniform angular velocity \( \Omega \) with respect to some inertial frame (see Figure E2.6).

(a) What is the configuration space for the system?
(b) Use the spatial angular velocity \( \omega \) of the ball to express the constraints of the system.
(c) Using the definition of the spatial angular velocity, turn the constraints from the previous part of the problem into a constraint of the type we discuss. Is the constraint linear? Why or why not?

(d) Write the Lagrangian for the system in terms of the spatial angular velocities.

(e) Use the previous part of the problem, give an expression for the Lagrangian as a function on the tangent bundle of the configuration space. Use Exercise E1.11(e) to make for a nicer expression of the rotational kinetic energy.

You will observe that this Lagrangian is not altogether a pleasant one, and that writing the Euler-Lagrange equations is not a pleasing prospect.

(f) However, use the fact that the Euler-Lagrange equations are equivalent to Newton's equations to obtain the equations of motion in terms of the spatial angular velocities and the position of contact of the ball with the table.

E2.37 Show that the body angular velocity of a rigid body is not a vector in the tangent space to the configuration space \( Q = SO(3) \).

E2.38 Let \( Q \) be a configuration space and consider a function \( H : \mathbb{R} \times T^*Q \to \mathbb{R} \). In a coordinate chart, define a map \( FH : \mathbb{R} \times T^*Q \to \mathbb{R} \times TQ \) by

\[
FH(t, q, p) = \left( q, \frac{\partial H}{\partial p} \right).
\]

(a) Show that \( FH \) is well-defined, i.e., that the above definition is independent of coordinates, and that \( FH(t, \alpha) \in T_qQ \).

Call the function \( H \) hyperregular if \( FH \) is a diffeomorphism. For a hyperregular \( H \), define a function \( LH : \mathbb{R} \times TQ \to \mathbb{R} \) by

\[
LH(t, v_q) = \langle FH^{-1}(v_q); v_q \rangle - H(t, FH^{-1}(v_q)).
\]

(b) State and prove the natural result corresponding to Theorem 2.9.2.

(c) Show that for a hyperregular Lagrangian, \( LH \) is a hyperregular Hamiltonian and that \( FH^{-1} = FH_L \).

(d) Show that for a hyperregular Hamiltonian, \( LH \) is a hyperregular Lagrangian and that \( FH^{-1} = FL_H \).