Chapter 4

An introduction to control theory for Lagrangian systems

In this chapter we undertake rather a hedgepodge of control theoretic topics. The systematic investigation of control theory for mechanical systems is in its infancy, so one cannot expect much in the way of unity here. What’s more, what is known is often beyond the scope of our presentation. What we try to do is indicate how the special structure of Lagrangian systems plays out when one employs some of the standard control theoretic techniques.

The most pleasure is apt to be derived from the material in this section by those who have some background in control. However, much of the material is self-contained. An excellent introduction to the mathematical foundations of control is the book by Sonntag [1998]. We refer to Spong and Vidyasagar [1989] for more detail on the material in Section 4.2 on robot control. In Section 4.4 it would be helpful if the reader had some background in linear systems theory, and a good source for this is [Broggett 1970]. The subject of passivity methods is dealt with in the book (Ortega, Loria, Nicklasson, and Sira-Ramirez [1998]). We should bluntly state that the reader who forms an opinion about the subject matter in control theory based upon the material in this chapter will have a rather non-conventional view of the subject.

4.1 The notion of a Lagrangian control system

In this section we shall simply describe a general control problem in Lagrangian mechanics. Our setting here will be far too general for us to be able to say much about it, but it forms the basis for the specialising as we do in subsequent sections.

Let $L$ be a Lagrangian on a configuration space $Q$. We consider a collection $\mathcal{F} = \{F^1, \ldots, F^m\}$ of forces on $Q$. The idea is that each one of these forces represents a direction in which we have control over our system. The total force which we command is then a linear combination of the forces $F^1, \ldots, F^m$. Following the convention in control theory, we denote the coefficients in this linear combination by $u_1, \ldots, u_m$. Thus the governing differential equations in a set of coordinates are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = u_i F^i, \quad i = 1, \ldots, n. \quad (4.1)$$

We shall call a pair $(L, \mathcal{F})$ as above a Lagrangian control system. Let us for simplicity make the assumption that the subspace of $T_qQ$ spanned by $F^1(t, v_q), \ldots, F^m(t, v_q)$ is independent of $q$. If for each $(t, v_q) \in \mathbb{R} \times TQ$ we have

$$\text{span}_\mathbb{R} \{ F^1(t, v_q), \ldots, F^m(t, v_q) \} = T_qQ,$$

we say that the system is fully actuated. Otherwise we say that it is underactuated. Generally, the issues surrounding the control of fully actuated systems are comparatively straightforward. Underactuated systems are quite challenging, on the other hand.

When one looks at equation (4.1) as a control system, the types of questions one is interested in differ from those one considers when one is interested in dynamics. One wishes, in control, to design the coefficients $u_1, \ldots, u_m$ so that the system behaves in a desired manner. If one designs the controls to be functions only of the time, $u_1(t), \ldots, u_m(t)$, one says the controls are open loop. The idea with open loop control is that you have analysed your problem and determined that the specified controls, when substituted into (4.1) will accomplish the desired objective. The problem is that this procedure is often not robust. It will suffer in the presence of model inaccuracies, uncertainty in initial conditions, etc. Thus one often wishes to have the controls somehow loop track of the current state using feedback, as well as perhaps the time, say, $u_1(t, v_q), \ldots, u_m(t, v_q)$, so that if one is not quite doing what is expected, then the control can compensate. This is called a closed loop control.

Some examples of control problems are as follows.

1. The reconfiguration problem: Here one wishes to start the system at time $t = 0$ at a state $v_0 \in TQ$ and design the controls $u_1, \ldots, u_m$ so that at time $t = T > 0$ the system, upon solving the equation (4.1), will arrive at the state $v_T$.

2. The trajectory tracking problem: For this problem, one has a prescribed curve $t \mapsto v_{\text{des}}(t) \in Q$ one wishes to follow. The objective is to design the controls so that one follows this curve, or follows it as closely as possible.

3. The point stabilisation problem: Here one has a point $q_0 \in Q$ and one wishes to design the controls $u_1, \ldots, u_m$, probably closed loop, so that $q_0$ is an asymptotically stable equilibrium point for the equations (4.1).

Sometimes one will also have outputs for the system. These will be in the form of functions $h^\alpha : \mathbb{R} \times TQ \to \mathbb{R}, \alpha = 1, \ldots, s$. In such cases, one becomes not so much interested in the states in $TQ$ as in the value of the outputs. Thus to the above problems, one may wish to add the following.

4. The output tracking problem: The problem here is something like the trajectory tracking problem, except that one is interested in making the outputs follow a desired path. Thus one has $s$ desired functions of time, $h_{1}^{\text{des}}, \ldots, h_{s}^{\text{des}}$, and one wishes to design the controls $u_1, \ldots, u_m$ so that the output follows these desired outputs, or at least does so as nearly as possible.

5. The output stabilisation problem: Here one has a fixed operating point for the output, say $h_0 \in \mathbb{R}$, and one wishes to design controls in such a manner that outputs which start near $h_0$ end up at $h_0$ as $t \to \infty$. Thus $h_0$ becomes an asymptotically stable fixed point, in some sense.

4.2 “Robot control”

We place the title of this section in inverted commas because the subject of robot control is large, and there are many issues involved. We will only address the most basic ones. For a more thorough overview, see Chapter 4 of [Murray, Li, and Sastry 1994], or the book [Spong and Vidyasagar 1989].

Before we begin with the control theory proper, let us state what we mean by a system to which we may apply “robot control.” This is a rather limited class of problems. To be
precise, a Lagrangian control system \((L, \mathcal{F})\) on \(Q\) is a **robotic control system** when

1. \(Q = \mathbb{R}^n\), and the canonical coordinate chart has preference over all others,
2. \(L\) is time-independent and hyperregular,
3. \((L, \mathcal{F})\) is fully actuated, and
4. unlimited force to the actuators is available.

Of course, the statement that the canonical coordinate chart is “preferred” is rather nebulous. Its presence as an assumption is to justify certain coordinate-dependent constructions made during the ensuing discussion.

It is perhaps useful to give some examples of robotic control systems, to ensure that our assumptions include what one might certainly wish to include in this class. First of all, during the ensuing discussion. Of course, the statement that the canonical coordinate chart is “preferred” is rather nebulous. Its presence as an assumption is to justify certain coordinate-dependent constructions made during the ensuing discussion.

The equations of motion for a robotic control system

4.2.1 The equations of motion for a robotic control system

Let us use the assumption that the canonical coordinate chart is preferred. The Euler-Lagrange equations are then

\[
\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \dot{x}^j + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} \dot{x}^j - \frac{\partial L}{\partial x^j} = u_i, \quad i = 1, \ldots, n.
\]

If we define a symmetric invertible matrix function \(M : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\) to have components \(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}, i, j = 1, \ldots, n\), and if we define a map \(N : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) by

\[
N(x, v)_i = \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} \dot{x}^j - \frac{\partial L}{\partial x^j}, \quad i = 1, \ldots, n,
\]

then the equations of motion can be written

\[
M(x, \dot{x}) \ddot{x} + N(x, \dot{x}) = u.
\] (4.2)

Let us specialise the notation further to the situation when the Lagrangian \(L\) is derived from a simple mechanical system \((Q, g, V)\), which will most often be the case. In this case one readily sees, first of all, that \(M\) is independent of \(v\), and indeed is simply the matrix of components for the Riemannian metric \(g\) in the standard coordinates for \(\mathbb{R}^n\). The term \(N\) can also be effectively broken down. We define a map \(C : \mathbb{R}^n \rightarrow L(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)\) by

\[
C(x)(v_1, v_2)_i = \frac{1}{2} \left( \frac{\partial M_{ij}}{\partial x^k} \dot{x}^k + \frac{\partial M_{ik}}{\partial x^j} \dot{x}^j - \frac{\partial M_{jk}}{\partial x^i} \dot{x}^k \right) v_1^j v_2^k.
\]

Thus \(C\) is related in a simple way to the Christoffel symbols for the Riemannian metric \(g\) in the standard coordinates. If \(dV\) is the vector of partial derivatives of the potential function \(V\), then the equations (4.2) specialise to the equations

\[
M(x) \ddot{x} + C(x)(\dot{x}, \dot{x}) + dV = u.
\] (4.3)

The term \(C(x)(\dot{x}, \dot{x})\) is often given the name “Coriolis forces.” Note that we violate our code of coordinate invariance in writing these equations. Indeed, in Remark 2.4.12 we explicitly advised against writing equations in the form (4.3)! However, we justify what we have done by saying that we have used our assumption 1 for robotic control systems where the standard coordinates for \(Q\) are preferred. Also, since we are supposed to be talking about robot control, we may as well do as most robot control folks do, and write the equations as (4.2) or (4.3).

Before we move on, let us give a useful property of the notation used in (4.3).

4.2.1 Lemma With \(M\) and \(C\) as used in (4.3) we have \(\dot{x}^i M_{ij} \ddot{x} - 2 \dot{x}^i (C(x)(\dot{x}, \dot{x})) = 0\).

**Proof** We compute

\[
M_{ij} = \frac{\partial M_{ik}}{\partial x^j} \dot{x}^k.
\]

It is then a straightforward computation using the definition of \(C\) to verify that

\[
\dot{x}^i M_{ij} \ddot{x} - 2 \dot{x}^i (C(x)(\dot{x}, \dot{x})) = \left( \frac{\partial M_{jk}}{\partial x^i} \dot{x}^j - \frac{\partial M_{ik}}{\partial x^j} \dot{x}^j \right) \dot{x}^k \dot{x}^j.
\]

Since the expression in the brackets is skew-symmetric with respect to the indices \(i\) and \(j\), the result follows.

Lewis [1997] gives an interpretation of this result in terms of conservation of energy.
4.2.2 Feedback linearisation for robotic systems

The title of this section is a bit pretentious. What we talk about in this section is sometimes referred to as the “computed torque” control law. It is, however, an example of a more general technique known as feedback linearisation, and this is why the title is as it is.

The setup for the computed torque control law, like that for the PD control law in the next section, is that one has a robotic control system $(L, \mathcal{F}_{ran})$ on $Q$, and a desired trajectory $t \mapsto x_{\text{des}}(t)$ for the configuration of the system. If the actual robot is evolving according to a curve $t \mapsto x(t)$ in configuration space, the error $t \mapsto e(t)$, is then the difference between where one is and where one wants to be:

$$e(t) = x(t) - x_{\text{des}}(t).$$

The idea is that one wishes to design a control law $u$ for (4.2) so that the error goes to zero as $t \to \infty$. The computed torque control law is defined by

$$u_{\text{CT}} = M(x, \dot{x})\ddot{x}_{\text{des}} + N(x, \dot{x}) - M(x, \dot{x})(K_v \dot{e} + K_p e),$$

where $K_v, K_p \in \mathbb{R}^{n \times n}$ are matrices, designed so as to accomplish a stated objective. Let us first get the form of the error dynamics using the computed torque control law.

4.2.2 Proposition Let $t \mapsto x_{\text{des}}(t)$ be a curve in $\mathbb{R}^n$. With $u(t) = u_{\text{CT}}(t)$ in (4.2), the error $e(t) = x(t) - x_{\text{des}}(t)$ satisfies the differential equation

$$\ddot{e}(t) + K_v \dot{e}(t) + K_p e(t) = 0.$$

Proof Substituting $u_{\text{CT}}$ into (4.2), some straightforward manipulation yields

$$M(x, \dot{x})\ddot{e}(t) + K_v \dot{e}(t) + K_p e(t) = 0.$$}

The result follows since $M(x, v)$ is invertible for all $v$ and $v$.

Now it is a simple matter to give a general form for $K_v$ and $K_p$ so that the error dynamics go to zero at $t \to \infty$.

4.2.3 Proposition If $K_v$ and $K_p$ are symmetric and positive-definite, then the error dynamics in Proposition 4.2.2 satisfy $\lim_{t \to \infty} e(t) = 0$.

Proof Writing the differential equation for the error in first-order form gives

$$\begin{bmatrix} \dot{e} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} 0 & I_v \\ -K_p & K_v \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix},$$

where $f = \dot{e}$. The result will follow if we can show that all eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I_v \\ -K_p & K_v \end{bmatrix}$$

have negative real part. Let $\lambda \in \mathbb{C}$ be an eigenvalue with eigenvector $(v_1, v_2) \in \mathbb{C}^n \times \mathbb{C}^n$. We must then have

$$\begin{bmatrix} v_2 \\ -K_p v_1 - K_v v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$: $\langle z_1, z_2 \rangle = \bar{z}_1^T z_2$, where $\bar{\cdot}$ means complex conjugation. We then have

$$\langle \lambda x_1, v_1 \rangle = \langle \bar{\lambda} x_1, v_2 \rangle = \langle v_1, \lambda v_2 \rangle = \langle v_1, -K_p v_1 - K_v v_2 \rangle = -\langle v_1, K_p v_1 \rangle - \lambda \langle v_1, K_v v_1 \rangle.$$

We also have $\langle \lambda x_1, v_1 \rangle = \lambda^2 \langle v_1, v_1 \rangle$, thus giving us

$$\langle v_1, v_1 \rangle \lambda^2 + \langle v_1, K_p v_1 \rangle \lambda + \langle v_1, K_v v_1 \rangle = 0.$$

Now note that if $v_1 = 0$ then we also have $v_2 = 0$. Since eigenvectors are necessarily nonzero, this implies that $v_1 \neq 0$. Thus $\lambda$ satisfies the equation

$$a\lambda^2 + b\lambda + c = 0,$$

where $a > 0$ since $v_1 \neq 0$, and where $b, c > 0$ since $K_p$ and $K_v$ are symmetric and positive-definite. We therefore conclude that $\lambda$ has strictly negative real part, as desired.

The computed torque control law has the advantage, therefore, of being able to track accurately any desired reference trajectory. To do so, however, may require significant actuator forces. Nevertheless, it is an effective control strategy.

4.2.4 PD control

It is possible to use some simple ideas from classical control for robotic systems, and perhaps the simplest controller which might be effective is PD control, or Proportional-Derivative control. Here one simply defines the control to be the sum of two terms, one proportional to the error, and one proportional to the derivative of the error. That is, for a given desired trajectory $t \mapsto x_{\text{des}}(t)$ in $Q$, we define

$$u_{\text{PD}}(t) = -K_v \dot{e}(t) - K_p e(t),$$

for symmetric positive-definite matrices $K_v, K_p \in \mathbb{R}^{n \times n}$, and where $e(t) = x(t) - x_{\text{des}}(t)$, as before. The PD control law is not as effective as the computed torque controller for following arbitrary reference trajectories. This is a general failing of PD controllers, often alleviated by implementing integral control. However, the following result indicates that a PD controller will stabilise a robotic control system to a desired configuration when the Lagrangian is derived from a simple mechanical system. Since almost all robotic systems are in practice of this form, this is not at all a significant restriction.

4.2.4 Proposition Let $(L, \mathcal{F}_{ran})$ be a robotic control system with $L$ the Lagrangian for the simple mechanical system $(Q, g, 0)$ with zero potential energy. If $t \mapsto x_{\text{des}}(t)$ is a constant desired trajectory, then taking $u(t) = u_{\text{PD}}(t)$ in (4.3) gives $\lim_{t \to \infty} e(t) = 0$.

Proof We will use the LaSalle Invariance Principle to show that $x_{\text{des}}$ is an asymptotically stable equilibrium point for (4.3) if we take $u = u_{\text{PD}}$. First note that the equations (4.3) with $u = u_{\text{PD}}$ are

$$M(x)\ddot{x} + C(x)(\dot{x}, \dot{x}) + K_v \dot{x} + K_p(x - x_0) = 0.$$ (4.4)
Now define a Liapunov function on $Q$ by
\[ G(x, v) = \frac{1}{2} v^T M(x) v + \frac{1}{2} (x' - x_0) K_p (x - x_0). \]
We compute
\[
\dot{G}(x, \dot{x}) = \dot{x}^T M(x) \dot{x} + \frac{1}{2} \dot{x}^T \dot{M}(x) \dot{x} + x^T K_p (x - x_0) \\
= -\dot{x} (C(x)(\dot{x}, x)) + \frac{1}{2} \dot{x}^T \dot{M}(x) \dot{x} - \dot{x}^T K_v \dot{x} \\
= -\dot{x}^T K_v \dot{x}.
\]
Here we have used Lemma 4.2.1. Note that the closed-loop equations (4.4) are forced Euler-Lagrange equations with Lagrangian
\[ \dot{L}(x, v) = L(x, v) - \frac{1}{2} (x' - x_0^T) K_p (x - x_0). \]
and with external force $F(x, v) = K_v v$. This force is strictly dissipative so we apply the LaSalle Invariance Principle. The LaSalle Invariance Principle states that the system will tend towards the set where $F(x, v) \cdot v = 0$. However, this set is comprised of those points in $TQ$ where $v = 0$. Now we note, again with the LaSalle Invariance Principle in mind, that the largest invariant set for $L$ for which $v = 0$ is the set of critical points for $L$ of the form $(x, 0)$. However, for such critical points $x$ must be a critical point for the potential function
\[ \tilde{V}(x) = \frac{1}{2} (x' - x_0^T) K_p (x - x_0). \]
Since $x_0$ is the only critical point for $\tilde{V}$, the result follows from Corollary 3.5.7. 

**Murray, Li, and Sastry [1994] provide an extension of the PD control we have defined here which allows one to asymptotically track general reference trajectories. Also, the ideas behind what we suggest here can be extended in a geometric framework which alleviates some of the inherent restrictions of the PD control we suggest in this section. We refer the reader to [Bullo and Murray 1999, Koditschek 1989] for ideas along these lines.**

### 4.3 Passivity methods

Sorry, not this year. See [Ortega, Loria, Nicklasson, and Sira-Ramirez 1998].

**4.4 Linearisation of Lagrangian control systems**

The material in the previous section was geared towards fully actuated systems, and a special class of these, even. When confronted with a system with fewer controls than degrees of freedom, things become more difficult. For certain control problems, notably the point stabilisation problem, a good thing to try first is linearisation, just as when studying some of the inherent restrictions of the PD control we suggest in this section. We refer the reader here which allows one to asymptotically track general reference trajectories. Also, the ideas tend towards the set where $0 = TQ$.

**4.4.1 The linearised system**

When linearising (4.1), one wishes to linearise both with respect to control and state. In Section 3.2 we already linearised the left hand side of (4.1), so it only remains to linearise the terms with the control. This, however, is trivial since this term is already linear in the control. Therefore, when linearising, one need only substitute the value of the equilibrium point into the forces. With this as motivation, we make the following definition.

**4.4.1 Definition**

Let $(L, \mathcal{F})$ be a Lagrangian control system on $Q$ and let $q_0$ be an equilibrium point for $L$. The **linearisation** of $(L, \mathcal{F})$ is the quadruple $(M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))$ where $M_L(q_0), C_L(q_0),$ and $K_L(q_0)$ are as defined in Proposition 3.2.10, and where $\mathcal{F}(q_0) = \{ F^1(0_0), \ldots, F^m(0_0) \} \subset T_{q_0}Q$. 

Corresponding to the linearisation is the following linear control system on $T_0Q$:
\[ M_L(q_0) \dot{q}(t) + C_L(q_0) \dot{q}(t) + K_L(q_0) q(t) = u \cdot F^m(0_0), \]
where $t \mapsto q(t)$ is a curve on $T_0Q$. (We make a nasty abuse of notation here, so beware: we let $q$ denote both points in $Q$ and points in $T_0Q$.) In the same way as we did when linearising near equilibria in studying dynamics, let us suppose that $L$ is hyperregular so that $M^T_L(q_0)$ is invertible. In this case, let us define a linear map $B_L(q_0): \mathbb{R}^m \to T_{q_0}Q$ by
\[ B_L(q_0)(u_1, \ldots, u_m) = u_1 M^T_L(q_0)(F^1(0_0)) + \cdots + u_m M^T_L(q_0)(F^m(0_0)). \]
We then arrive at the linear control system on $T_0Q \oplus T_{q_0}Q$ given by
\[
\begin{pmatrix}
\dot{q} \\
\dot{u}
\end{pmatrix} = 
\begin{pmatrix}
0 & id_V \\
-M^T_L(q_0) C_L^T(q_0) & -M^T_L(q_0) C_L^T(q_0)
\end{pmatrix}
\begin{pmatrix}
q \\
u
\end{pmatrix}
+ B_L(q_0) u.
\]
This is a linear control system in the best tradition. That is to say, it is a time-invariant system of the form
\[ \dot{x} = Ax + Bu, \]
where
\[ x = \begin{pmatrix} q \\ u \end{pmatrix}, \]
\[ A = A_L(q_0) = \begin{pmatrix} 0 & id_V \\ -M^T_L(q_0) C_L^T(q_0) & -M^T_L(q_0) C_L^T(q_0) \end{pmatrix}, \]
\[ B = B_L(q_0) = \begin{pmatrix} 0 \\ B_L(q_0) \end{pmatrix}. \]

There is an exorbitant amount of literature on such systems [e.g., Brockett 1970], and we only address a few of the more basic notions for such systems. Our emphasis, like it was when we were dealing with dynamics, is on deciding the relationship between the linearised system and the actual system near the equilibrium point in question.

**4.4.2 Controllability of the linearised system**

A first basic question deals with controllability. The reader will recall that a linear control system
\[ \dot{x} = Ax + Bu, \]
where $t \mapsto x(t)$ is a curve in a vector space $V$, is **controllable** if for any $x_1, x_2 \in V$ there exists a control $u: [0, T] \to \mathbb{R}^m$ so that if the initial condition is $x(0) = x_1$, then $x(T) = x_2$. 

...
In brief, the system is controllable if it can be steered from any state to any other state. The 
Kalman rank condition says that the system is controllable if and only if the matrix
\[
\begin{bmatrix}
B & AB & \cdots & A^{N-1}B
\end{bmatrix}
\]
has rank equal to \( N = \dim(V) \), where \( A \in \mathbb{R}^{N \times N} \) and \( B \in \mathbb{R}^{N \times m} \) are the matrix representations of \( A \) and \( B \) in some basis for \( V \).

Let us see how this transfers to our Lagrangian setting. The first result we state is the most
general, and to state it requires some notation. Let \( \mathcal{I} \) be the collection of multi-
indices in \( \{1,2\} \). Thus an element \( I \in \mathcal{I} \) has the form \( I = (i_1, \ldots, i_n) \) where \( i_a \in \{1,2\} \) for \( a = 1, \ldots, k \). For a multi-index \( I \), we write \(|I|\) for its length. Thus \(|(i_1, \ldots, i_n)| = k \). For a multi-index \( I \in \mathcal{I} \) we denote by \(|I|_1 \) the number of 1’s in \( I \), and by \(|I|_2 \) the number of 2’s in \( I \). Note that \(|I|_1 + |I|_2 = |I| \).

Now suppose that we choose a basis for \( T_qM \) and denote by \( A_1(q_0), A_2(q_0) \in \mathbb{R}^{n \times n} \) the matrices of \( M_1(q_0) = K_1(q_0) \) and \( M^2(q_0) = C_2(q_0) \), respectively, with respect to this basis. Now for \( k \in \mathbb{Z}_+ \) define
\[
\mathbb{R}^{n \times n} \ni M^{(k)}(q_0) = \begin{cases}
\sum_{j=k/2}^k \sum_{|I|=j} A_1(q_0) \cdots A_j(q_0), & k \text{ even} \\
\sum_{j=(k+1)/2}^k \sum_{|I|=j} A_1(q_0) \cdots A_j(q_0), & k \text{ odd}.
\end{cases}
\]

Let us give the first few of the matrices \( M^{(k)}(q_0) \) so that the reader might begin to understand
the cumbersome notation. We determine that
\[
\begin{align*}
M^{(1)}(q_0) &= A_2(q_0), \\
M^{(2)}(q_0) &= A_1(q_0) + A_2(q_0)^2, \\
M^{(3)}(q_0) &= A_1(q_0)A_2(q_0) + A_1(q_0)A_1(q_0) + A_2(q_0)^3, \\
M^{(4)}(q_0) &= A_1(q_0)^2 + A_1(q_0)A_2(q_0)^2 + A_2(q_0)A_1(q_0)A_2(q_0) + A_2(q_0)^2A_1 + A_2(q_0)^3.
\end{align*}
\]

Hopefully this is enough to enable the reader to see how one proceeds, using the definitions for
\( M^{(k)}(q_0), k > 0 \).

With this notation in hand, we may state our main result concerning controllability of
linearisations.

4.4.2 Theorem Let \((L, \mathcal{F})\) be a Lagrangian control system on \( Q \) with \( L \) a hyperregular
Lagrangian. Let \( q_0 \) be an equilibrium point for \( L \) and let \((M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))\) be
the linearisation of \((L, \mathcal{F})\) at \( q_0 \). The linearised system
\[ \dot{x} = A_L(q_0)x + B_F(q_0)u \]
is controllable if and only if the matrix
\[
\begin{bmatrix}
B(q_0) & M^{(1)}(q_0)B(q_0) & \cdots & M^{(2n-2)}(q_0)B(q_0)
\end{bmatrix}
\]
has rank \( 2n \), where \( M^{(k)}(q_0), k = 1, \ldots, 2n-2 \) are as defined above with respect to some
basis for \( T_{q_0}Q \), and where \( B(q_0) \) is the matrix of \( B_L(q_0) \) with respect to the same basis.

Proof The following lemma contains the bulk of the idea of the tedious proof.

1 Lemma Let \( A \in \mathbb{R}^{2n \times 2n} \) and \( B \in \mathbb{R}^{2n \times m} \) be given by
\[
A = \begin{bmatrix}
0 & I_n \\
A_1 & A_2
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
B_2
\end{bmatrix},
\]
for \( A_1, A_2 \in \mathbb{R}^{n \times n} \) and \( B_2 \in \mathbb{R}^{n \times m} \). Let \( M^{(0)} = I_n \) and for \( k \in \mathbb{Z}_+ \) define
\[
M^{(k)} = \begin{cases}
\sum_{j=k/2}^k \sum_{|I|=j} A_1 \cdots A_j, & k \text{ even} \\
\sum_{j=(k+1)/2}^k \sum_{|I|=j} A_1 \cdots A_j, & k \text{ odd}.
\end{cases}
\]

Then, for \( k \in \mathbb{Z}_+ \) we have
\[
A^kB = \begin{cases}
M^{(k-1)}B_2, & k \text{ even} \\
M^{(k)}B_2 = (A_1)^kB_2, & k \text{ odd}.
\end{cases}
\]

One readily verifies now that the lemma holds for \( k = 1 \). Suppose it true for \( k > 1 \). We
then have
\[
A^{k+1}B = \begin{cases}
M^{(k)}B_2, & \text{if } k \text{ is even} \\
M^{(k+1)}B_2 = (A_1)^k + A_1M^{(k-1)}B_2 + A_2M^{(k)}B_2, & \text{if } k \text{ is odd}.
\end{cases}
\]

First let us suppose that \( k \) is even. In this case we have
\[
A_1M^{(k-1)} + A_2M^{(k)} = \sum_{j=k/2}^k \sum_{|I|=j} A_3A_4 \cdots A_j + \sum_{j=k/2}^k \sum_{|I|=j} A_2A_3 \cdots A_j.
\]

A similarly styled computation also shows that \( A_1M^{(k-1)} + A_2M^{(k)} = M^{(k+1)} \) when \( k \) is
odd. This now gives the result.

\( \blacktriangledown \)
4.4 Linearisation of Lagrangian control systems

To proceed with the proof of the theorem, note that the situation dealt with in the theorem is just as in the lemma with \( A_1 = -A_1(q_0) \), \( A_2 = -A_2(q_0) \), and \( B_2 = B(q_0) \). Furthermore, by the lemma we have

\[
\begin{bmatrix} B & AB & A^2B & \cdots & A^{2n-1}B \end{bmatrix} = \begin{bmatrix} 0 & B_2 & M^{(1)}B_2 & M^{(2)}B_2 & \cdots & M^{(2n-2)}B_2 \end{bmatrix}.
\]

Let us define \( M_1, M_2 \subset \mathbb{R}^{2n \times 2nm} \) by

\[
M_1 = \begin{bmatrix} 0 & B_2 & M^{(1)}B_2 & M^{(2)}B_2 & \cdots & M^{(2n-2)}B_2 \end{bmatrix},
\]

\[
M_2 = \begin{bmatrix} B_2 & M^{(1)}B_2 & M^{(2)}B_2 & \cdots & M^{(2n-2)}B_2 \end{bmatrix}.
\]

Note \( \text{image}(M_1) \subset \text{image}(M_2) \). Therefore the rank of

\[
\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}
\]

will equal \( 2n \) if and only if \( \text{rank}(M_1) = n \), as stated. \[\square\]

Let us now specialise the above theorem to the important case when \( C_L(q_0) = 0 \). In this case, with our above notation with respect to a basis for \( T_{q_0}Q \), we have \( A_1(q_0) = 0 \). We now show, using Theorem 4.4.2, that the following condition for controllability of the linearised system holds.

**4.4.3 Corollary** Let \((L, \mathcal{F})\) be a Lagrangian control system on \( Q \) with \( L \) a hyperregular Lagrangian. Let \( q_0 \) be an equilibrium point for \( L \) and let \((M_1(q_0), C_1(q_0), K_1(q_0), \mathcal{F}(q_0))\) be the linearisation of \((L, \mathcal{F})\) at \( q_0 \), supposing that \( C_L(q_0) = 0 \). The linearised system

\[
\dot{x} = A_1(q_0)x + B_2(q_0)u
\]

is controllable if and only if the matrix

\[
\begin{bmatrix} B(q_0) & A_1(q_0)B(q_0) & \cdots & A_1^{n-1}(q_0)B(q_0) \end{bmatrix}
\]

has rank \( n \), where \( A_2(q_0) \) is the matrix of \( M^{(1)}_L(q_0) - K_2^L(q_0) \) with respect to some basis for \( T_{q_0}Q \), and where \( B(q_0) \) is the matrix of \( B_2(q_0) \) with respect to the same basis.

**Proof** One verifies using the definition of \( M^{(k)}_L(q_0) \) above, with \( A_2(q_0) = 0 \), that \( M^{(k)}_L(q_0) = \begin{cases} 0, & k \text{ odd} \\ \left(A_1(q_0)^k \right)^{2/3}, & k \text{ even}. \end{cases} \)

The result now follows directly from Theorem 4.4.2. \[\square\]

**4.4.4 Remark** Corollary 4.4.3 holds in the important case where \( q_0 \) is an equilibrium point for a simple mechanical system since, by Proposition 3.3.7, \( C_L(q_0) = 0 \) at equilibrium points for simple mechanical systems. Also note that the condition for controllability of the linearised system simplifies significantly in this case since the matrix whose rank needs to be checked is essentially half the size. \[\square\]

Let us look at a simple example on the basis of the previous remark.

---

4.4.5 Example (Example 2.6.3 cont’d) We consider the pendulum of Example 2.6.3, and now apply a torque at the base of the pendulum. One might intuitively expect the system to be controllable, but let us perform the linearisation analysis, just so we know how it goes.

The Lagrangian for the system we take to be \( L(q, \dot{q}) = m\ell^2 q'' + m\ell q \sin \theta \). Note that we reinset units into the problem to make it a little more interesting. The Euler-Lagrange equations, with the torque at the base of the pendulum denoted by \( u \), read

\[ m\ell q'' + m\ell q \sin \theta = u. \]

There are two equilibrium points, \( q_1 = 0 \) and \( q_2 = \pi \). We compute the linearisations of \( L \) at these points to be

\[
\begin{align*}
M_L(q_1) &= m\ell^2, & K_L(q_1) &= m\ell q_1 \\
M_L(q_2) &= m\ell^2, & K_L(q_2) &= -m\ell q_2.
\end{align*}
\]

with \( C_L(q_1) = C_L(q_2) = 0 \) by virtue of the system being a simple mechanical system. We then determine that

\[
M_L^2(q_1) - K_L^2(q_1) = \frac{q_1}{\ell}, \quad M_L^2(q_2) + K_L^2(q_2) = -\frac{q_2}{\ell}.
\]

We also have \( B_L(q_1) = B_L(q_2) = \frac{1}{\ell^2} \). Now, adopting the notation of Corollary 4.4.3, we find that

\[
\begin{bmatrix} B(q_1) & A_1(q_1)B(q_1) & \cdots & A_1^{n-1}(q_1)B(q_1) \\ B(q_2) & A_1(q_2)B(q_2) & \cdots & A_1^{n-1}(q_2)B(q_2) \end{bmatrix} = \begin{bmatrix} 1 & \frac{q_1}{\ell} & -\frac{q_2}{\ell} \\ 1 & \frac{q_1}{\ell} & -\frac{q_2}{\ell} \end{bmatrix}.
\]

In each case, of course the rank of the matrix has maximal rank 1. This example is perhaps too trivial to illustrate much, and the reader is encouraged to try out the results on the more interesting examples in the exercises. \[\square\]

An important corollary is the following which deals with simple mechanical systems with no potential. Recall that for such systems, every point \( q_0 \in Q \) is an equilibrium point.

**4.4.6 Corollary** Let \((L, \mathcal{F})\) be a Lagrangian control system with \( L \) the Lagrangian for a simple mechanical system \((Q, g, 0)\) with zero potential. For any \( q_0 \in Q \), the linearisation \((M_L(q_0), C_L(q_0), K_L(q_0), \mathcal{F}(q_0))\) is controllable if and only if \((L, \mathcal{F})\) is fully actuated.

**Proof** In the notation of Theorem 4.4.2, \( A_1(q_0) = A_2(q_0) = 0 \). Therefore, using Corollary 4.4.3 the system is controllable if and only if the matrix \( B(q_0) \) has rank \( n \). But this will happen if and only if

\[
\text{span}_{\mathbb{R}} \{ F^1(v_0), \ldots, F^m(v_0) \} = T_{q_0}Q.
\]

By our assumption of the independence of the dimension of the span of the forces \( F^1(v_q), \ldots, F^m(v_q) \) on \( v_q \in TQ \), the result follows. \[\square\]

This result has important implications. It says that one cannot expect anything helpful to occur when linearising a simple mechanical system with a kinetic energy Lagrangian, except in the trivial case when we have full authority available to us with the controls. In these cases, we may resort to the methods of Section 4.2. In other cases, when the system is underactuated, things become rather complicated, and we mention a few “simple” ideas in Section 4.5.
Let us give another important application of Theorem 4.4.2. Let us suppose that we have a system which is subjected to a dissipative force which is linear in velocity, as is the case with viscous friction. The force we consider is thus of the form

$$F(v_t) = -R_q(v_t)$$

where $R_q$ is a positive semi-definite quadratic form on $T_qQ$. Note that this is simply a generalisation of the viscous force given in a simple example in Example 2.5.9-1. With a dissipative force of this type, an equilibrium point for $L$ will still be an equilibrium point with the addition of the dissipative force since this force vanishes when velocity is zero. Therefore, we may still linearise about equilibria for $L$, even in the presence of this dissipative force. To determine the controllability of the linearisation, we need a comparatively straightforward modification of the notation used in Theorem 4.4.2. Let $A_1(q_0) = A_i(q_0)$ be as used in that theorem, and let $A_2(q_0)$ be the matrix for the linear map $M^T_{L}(q_0) + C^2_{L}(q_0) + R^T_{q_0}$. Now define

$$\mathbb{R}^{n \times n} \ni \tilde{M}^{(k)}(q_0) = \begin{cases} \sum_{i=0}^{k} \sum_{|I| = k-i} A_i(q_0) \cdots A_{j}(q_0), & k \text{ even} \\ \sum_{i=(k+1)/2}^{k} \sum_{|I| = k-i} A_i(q_0) \cdots A_{j}(q_0), & k \text{ odd}. \end{cases}$$

The following result tells us when the resulting linearisation of a system with viscous dissipation is controllable.

4.4.7 Proposition Let $(L, F)$ be a Lagrangian control system on $Q$ with $L$ a hyperregular Lagrangian, and let $F$ be a dissipative force on $Q$ of the form $F(v_t) = R_q(v_t)$, with $R_q$ positive semi-definite, as above. If $q_0$ is an equilibrium point for $L$, let $(M_1(q_0), C_1(q_0), K_1(q_0), F(q_0))$ be the linearisation of $(L, F)$. Then the controllability of the system with the addition of the dissipative force $F$ is controllable if and only if the matrix

$$\begin{bmatrix} B(q_0) & M^{(1)}(q_0)B(q_0) & \cdots & M^{(2n-2)}(q_0)B(q_0) \end{bmatrix}$$

has rank $2n$, where $M^{(k)}(q_0)$, $k = 1, \ldots, 2n - 2$ are as defined above with respect to some basis for $T_qQ$, and where $B(q_0)$ is the matrix of $B_0(q_0)$ with respect to the same basis.

Proof We shall compute the linearisation of the system at the equilibrium point $q_0$. We do this by working in coordinates $(q^1, \ldots, q^n)$. Following the computations preceding Proposition 3.10.2, we Taylor expand about $(q_0, 0)$. The resulting expression is

$$\frac{\partial^2 L}{\partial q_i \partial q_j} \hat{q}_i(t) + \left( \frac{\partial^2 L}{\partial q_i \partial q_j} \right) \hat{q}_j(t) - \frac{\partial^2 L}{\partial q_i \partial q_j} \hat{q}_j(t) \cdots = -R_{q_i}(q_0) \hat{q}_i(t) + u_t F^\tau_{q_0}(q_0, 0), \quad i = 1, \ldots, n,$n

where $\hat{q}(t) = q'(t) - q_0$, $i = 1, \ldots, n$. Thus the linearised equations look like

$$\hat{q}_i(t) = q_i'(t) \cdots = R_{q_i}(q_0) \hat{q}_i(t) + u_t F^\tau_{q_0}(q_0, 0).$$

Here again we make the horrid abuse of notation of writing points in the tangent space $T_qQ$ as $q$. Using the fact that $L$ is hyperregular, we write these as first-order equations:

$$\begin{pmatrix} \frac{dq}{dt} \\ \frac{d\dot{q}}{dt} \end{pmatrix} = \begin{pmatrix} 0 & I_d \\ -M^T_{L}(q_0) + C^2_{L}(q_0) + R^T_{q_0} \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{u} \end{pmatrix} + \begin{pmatrix} 0 \\ B_0(q_0) \end{pmatrix} u.$$

The result now follows directly from Lemma 1 of the proof of Theorem 4.4.2.
4.5 Control when linearisation does not work

**control controllable** if for \( q_1, q_2 \in Q \) there exists a controlled trajectory \((c, u)\) defined on the interval \([0, T]\) with the property that \( c(0) = q_1 \) and \( c(T) = q_2 \). To determine the controllability conditions we let \( L(\mathcal{X}) \) be the smallest subset of vector fields on \( Q \) with the property that \( \mathcal{X} \subset L(\mathcal{X}) \) and with the property that \([X, Y] \in L(\mathcal{X})\) for all vector fields \( X, Y \in L(\mathcal{X})\). Now, for each \( q \in Q \), define a subspace \( L_0(\mathcal{X})_q \) by

\[
L_0(\mathcal{X})_q = \{ X(q) \mid X \in L(\mathcal{X}) \}.
\]

To compute \( L(\mathcal{X}) \) in practice, it often suffices to iterative take Lie brackets of the vector fields \( X_1, ..., X_s \) until one stops generating new directions. Thus one computes the vector fields \([X_a, X_b], [X_a, [X_a, X_c]], \ldots\), \( a, b, c = 1, \ldots, s \), and so on. Typically this process will “terminate” in the sense that \( L_0(\mathcal{X})_q \) will stop growing.

The following important result gives conditions on when a driftless system is controllable.

**Theorem 4.5.1 (Chow [1939])** A driftless control system \( \Sigma = (Q, \mathcal{X}) \) is controllable if \( \mathcal{X}(Q)_q = T_qQ \) for each \( q \in Q \). If the vector fields \( \mathcal{X} \) are real analytic, then this condition is also necessary.

Let us make some general comments concerning driftless systems.

**Remarks 4.5.2** We name driftless systems as we do because they are a specific case of control systems of the type

\[
c'(t) = X_0(c(t)) + u(t)X_a(c(t)),
\]

where we now have no control over the vector field \( X_0 \), which is called the drift vector field. For driftless systems, of course, the drift vector field is zero. Systems with nonzero drift vector fields are significantly harder to deal with. In particular, there are no known necessary and sufficient conditions for controllability of the type given in Theorem 4.5.1 for systems with drift.

2. Note that equation (4.5) essentially describes the set of curves \( t \mapsto c(t) \) whose tangent vectors lie in \( c'(t) \) lie in the subspace \( \text{span}_c \{ X_1(c(t)), \ldots, X_a(c(t)) \} \) for each \( t \). In this way, one can think of (4.5) as describing a linear constraint on \( Q \), exactly as we did in Section 2.6. With this interpretation, Theorem 4.5.1 gives conditions on when it is possible to connect any two points in \( Q \) with a curve satisfying the constraint.

3. If one linearises (4.5) about a point \( q_0 \), the resulting linear control system on \( T_{q_0}Q \) is simply

\[
\dot{q}(t) = B(q_0)u,
\]

where \( B(q_0) : \mathbb{R} \to T_{q_0}Q \) is defined by \( B(q_0)u = uX_a(q_0) \). The Kalman rank condition (the “A” matrix is zero) tells us that this linearisation is stable if and only if \( B(q_0) \) is surjective, i.e., if the tangent vectors \( X_1(q_0), \ldots, X_a(q_0) \) generate \( T_{q_0}Q \). Thus the system is controllable only in the trivial case where the inputs allow us to access all directions in \( Q \).

Now that we have dealt with the controllability question for driftless systems, let us look at how to handle some common control problems. First let us look at the problem of designing a control law which will stabilise the system to a desired point \( q_0 \). The following result states that the upshot of the observation Remark 4.5.2–3 is fatal as far as using feedback to stabilise a driftless system.
A controlled trajectory for an affine connection control system is a pair \((c, u)\) where \(u : [0, T] \rightarrow \mathbb{R}^m\) is piecewise differentiable, and where \(c : [0, T] \rightarrow Q\) has the property that the differential equation (4.6) is satisfied. The act of even defining appropriate notions of controllability for (4.6) requires some ideas we do not wish to deal with in full generality [see Lewis and Murray 1997]. Let us agree to deal only with the simplified notion of controllability called equilibrium controllability where the system possesses this property if for each \(q_1, q_2 \in Q\), there exists a controlled trajectory \((c, u)\) defined on the interval \([0, T]\) with the property that \(c'(0) = 0_{q_1}\) and \(c'(T) = 0_{q_2}\). Roughly, the system is equilibrium controllable when it can be steered from any point at rest to any other point at rest. As we have been doing all along, we shall assume that the dimension of the subspace \(Y_q = \text{span}_{\mathbb{R}} \{ Y_1(q), \ldots, Y_m(q) \}\) is independent of \(q\).

Note that (4.6) has a form somewhat similar to the equation (4.5) for driftless systems. However, the two types of systems are in no way the same! For example, the equations (4.6) are second-order, whereas the equations (4.5) are first-order. Despite the fact that the two types of equations are not equivalent, in some examples they are related in some way, and it is this fact which we will exploit in the remainder of the section.

4.5.3 Mechanical systems which are "reducible" to driftless systems It turns out that some, but certainly not all (in some sense very few indeed), mechanical systems can, in a limited sense, be thought of as driftless systems. Fortunately, these systems are ones for which linearisation is ineffective, so the connection with driftless systems provides an "in" to being able to do some control tasks for these systems.

The following definition establishes the type of correspondence we are after in this section.

4.5.4 Definition Let \(\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})\) be an affine connection control system and let \(\Sigma = (Q, \mathcal{X})\) be a driftless system. \(\Sigma_{\text{aff}}\) is reducible to \(\Sigma\) if the following two conditions hold:

(i) for each controlled trajectory \((c, \tilde{u})\) for \(\Sigma_{\text{aff}}\) defined on \([0, T]\) with \(\tilde{u}\) differentiable and piecewise \(C^2\), there exists a piecewise differentiable map \(u : [0, T] \rightarrow \mathbb{R}^m\) so that \((c, u)\) is a controlled trajectory for \(\Sigma_{\text{aff}}\);

(ii) for each controlled trajectory \((c, u)\) for \(\Sigma_{\text{aff}}\) defined on \([0, T]\) with \(c'(0) \in X_q\), there exists a differentiable and piecewise \(C^2\) map \(\tilde{u} : [0, T] \rightarrow \mathbb{R}^m\) so that \((c, \tilde{u})\) is a controlled trajectory for \(\Sigma\).

The idea of the definition is quite simple. It establishes that there is a correspondence between controlled trajectories for the driftless system and the affine connection control system. In making the correspondence, one has to be careful of two things.

1. The same types of controls cannot be used for both \(\Sigma_{\text{aff}}\) and \(\Sigma\). For the driftless system, if the input is discontinuous, this will imply that there will be instantaneous velocity jumps. Such phenomenon are not physically realisable for affine connection control systems since this would require infinite forces. This is because at points of an instantaneous velocity jump, acceleration will be infinite. This explains why in part (i) we need to add extra differentiability to the input for \(\Sigma\).

2. It will not be possible to assign to every controlled trajectory of \(\Sigma_{\text{aff}}\) a controlled trajectory of \(\Sigma\). This is clear since initial conditions for \(\Sigma_{\text{aff}}\) allow that \(c'(0)\) can be arbitrary, whereas all controlled trajectories for \(\Sigma\) will have \(c'(0) \in X_{\text{aff}}\). This explains the subsidiary condition on controlled trajectories for \(\Sigma_{\text{aff}}\) is part (ii) of Definition 4.5.4.

The following result gives the surprisingly simple answer to the question of when an affine connection control system is reducible to some driftless system.

4.5.5 Theorem (Lewis [1999]) An affine connection control system \(\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})\) is reducible to a driftless system \(\Sigma = (Q, \mathcal{X})\) if and only if the following two conditions hold:

(i) \(X_q = Y_q\) for each \(q \in Q\);

(ii) \(\nabla_q X(q) \in Y_q\) for every vector field \(X\) having the property that \(X(q) \in Y_q\) for every \(q \in Q\).

The first condition is perhaps not surprising, but neither is it obvious. It states that when establishing whether \(\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})\) is reducible to \(\Sigma = (Q, \mathcal{X})\) we may as well assume that \(\mathcal{X} = \mathcal{Y}\). The meat of the theorem is the second condition, however. This condition is one which is readily checked. It is also true that there will be very few systems which satisfy this condition. Nevertheless, there appears to be an unnaturally large class of physical systems which meet the criterion of Theorem 4.5.5, so it is an interesting one as concerns applications.

Not only does Theorem 4.5.5 provide conditions for when a system is reducible to a driftless system, but it turns out that when the conditions of the theorem are met, it is comparatively easy to derive the controls for the affine connection control system from those for the driftless system. To state the result, we need to introduce the symmetric product which takes two vector fields \(X\) and \(Y\) on \(Q\) and returns another vector field \(\langle X : Y \rangle\) defined by

\[
\langle X : Y \rangle = \nabla X Y + \nabla Y X.
\]

It is easy to show (see Exercise E4.4) that condition (ii) of Theorem 4.5.5 is equivalent to the statement that \(\langle X : Y \rangle(q) \in Y_q\) for every pair of vector fields \(X\) and \(Y\) with the property that \(X(q), Y(q) \in Y_q\). With this, we state the following result.

4.5.6 Proposition Let \(\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})\) be an affine connection control system which is reducible to the driftless system \((Q, \mathcal{X})\). Suppose that the vector fields \(\mathcal{X} = \{ Y_1, \ldots, Y_m \}\) are linearly independent and define \(\gamma_{ab} : Q \rightarrow \mathbb{R}\), \(a, b, d = 1, \ldots, m\), by

\[
(Y_a : Y_b) = \gamma_{ab} Y_d,
\]

which is possible by condition (ii) of Theorem 4.5.5. If \((c, \tilde{u})\) is a controlled trajectory for the driftless system \(\Sigma\) then, if we define the control \(u\) by

\[
u^d(t) = \tilde{u}^d(t) \tilde{u}^b(t)(\tilde{u}^d(t) + \frac{\nabla^d}{\gamma_{bd}}(c(t))), \quad d = 1, \ldots, m,
\]

\((c, u)\) is a controlled trajectory for the affine connection control system \(\Sigma_{\text{aff}}\).

Proof By definition \((c, \tilde{u})\) satisfy

\[
c'(t) = \tilde{u}^b(t) Y_a(c(t)).
\]
Therefore
\[
\nabla_{\xi(t)}X(t) = \nabla_{\xi(t)}(\tilde{u}^d(t)Y_d(c(t)))
\]
\[= \tilde{u}^d(t)\nabla_{\xi(t)}Y_d(c(t)) + \tilde{u}^d(t)\dot{Y}_d(c(t))
\]
\[= \tilde{u}^d(t)\nabla_{\xi(t)}Y_d(c(t)) + \tilde{u}^d(t)\dot{Y}_d(c(t))
\]
\[= \tilde{u}^d(t)\tilde{u}^d(t)\left(\nabla_{\xi(t)}Y_d(c(t)) + \dot{Y}_d(c(t))\right)
\]
\[= \tilde{u}^d(t)\tilde{u}^d(t)\left(\tilde{u}^d(t) + \frac{1}{2}T_d(c(t))\right)Y_d(c(t)).
\]

Now, if we define \( u \) as in the statement of the proposition, we have
\[
\nabla_{\xi(t)}X(t) = u^d(t)Y_d(c(t)),
\]
as desired.

Let us summarise the point of the development in this section.

4.5.7 Affine connection control systems reducible to a driftless system Suppose that an affine connection control system \( \Sigma_{\text{aff}} = (Q, \nabla, \mathcal{X}) \) is reducible to a driftless system \( \Sigma = (Q, \mathcal{X}). \) (Note that without loss of generality we may suppose the input vector fields are the same for \( \Sigma_{\text{aff}} \) and \( \Sigma \).) First examine the control problem to see if it is one for which the design may be done for the driftless system \( \Sigma \). Examples of control problems of this type are

1. steering from rest at a given configuration to another
2. stabilising to a point, provided that all initial conditions have velocities in \( \mathcal{Y} \).

One then does the design for the problem, if possible, using the driftless system, making use of the literature given in Section 4.5.1. Once the controls are found for the driftless system, then one uses Proposition 4.5.6 to translate controls for the driftless system to controls for the affine connection control system.

An example of an affine connection control system which can be reduced to a driftless system is given in Exercise E4.5.

4.5.8 Proposition (Bullo and Lynch [2001]) A vector field \( X \) is a decoupling vector field for an affine connection control system \( \Sigma_{\text{aff}} = (Q, \nabla, \mathcal{X}) \) if and only if

(i) \( X(q) \in \mathcal{Y}_q \) for each \( q \in Q \) and
(ii) \( \nabla_{X}X(q) \in \mathcal{Y}_q \) for each \( q \in Q \).

Proof First suppose that (i) and (ii) hold. To show that a vector field \( X \) is a decoupling vector field, it suffices to show that for any function \( f : Q \to \mathbb{R} \) and any integral curve \( c \) for \( fX \), there exists a control \( u \) so that \((c, u)\) is a controlled trajectory for \( \Sigma_{\text{aff}} \). Letting \( f \) and \( c \) be so chosen, we have
\[
\nabla_{\xi(t)}c(t) = \nabla_{\xi(t)}fX(c(t))
\]
\[= f^2(c(t))\nabla_{X}X(c(t)) + f(c(t))(\mathcal{L}_{X}f(c(t)))X(c(t)).
\]
Now using (i) and (ii) we have
\[
X(c(t)), \nabla_{X}X(c(t)) \in \mathcal{Y}_c(t).
\]
Therefore, there exists \( t \mapsto u(t) \) so that
\[
f^2(c(t))\nabla_{X}X(c(t)) + f(c(t))(\mathcal{L}_{X}f(c(t)))X(c(t)) = u^d(t)Y_d(c(t)).
\]

Now let us look at conditions which determine when a given vector field is a decoupling vector field.

4.5.9 Kinematically controllable systems Bullo and Lynch [2001] provide a notion which is weaker than that of equivalence to a driftless system we looked at in the last section. The notion they provide is applicable to a larger class of systems, and so is worth looking at as another means of approaching control for systems where linearisation methods are not applicable.

The idea of Bullo and Lynch is somewhat like the idea of reducibility presented in the previous section. There is a subtle difference, however, and this difference broadens the class of problems to which the methods can be applied, although the method of applicability is somewhat more restricted. The idea is that one starts with an affine connection control system \( \Sigma_{\text{aff}} = (Q, \nabla, \mathcal{X}) \) (one not necessarily reducible to a driftless system) and asks if there are any vector fields on \( Q \) whose integral curves can be followed up to arbitrary parameterisation. Let us be precise about this. A vector field \( X \) on \( Q \) is a **decoupling vector field** for \( \Sigma_{\text{aff}} \) if for every integral curve \( t \mapsto c(t) \) for \( X \) and for every reparameterisation \( t \mapsto \tau(t) \) for \( c \), there exists a control \( t \mapsto u(t) \) with the property that \((c, u)\) is a controlled trajectory. \( \Sigma_{\text{aff}} \) is **kinematically controllable** if there exists a collection of decoupling vector fields for \( \Sigma_{\text{aff}} \), \( \mathcal{X} = \{X_1, \ldots, X_r\} \), so that \( \mathcal{L}(\mathcal{X}) = T_qQ \) for each \( q \in Q \).

Before we provide the conditions for determining when a vector field is a decoupling vector field, let us say a few words about the implications of a system being kinematically controllable, and contrast it with the notion of reducibility introduced in the previous section. It is true that if a collection of vector fields \( \mathcal{X} = \{X_1, \ldots, X_r\} \) has the property that \( \mathcal{L}(\mathcal{X}) = T_qQ \) for each \( q \in Q \), then one is able to connect any two points in \( Q \) with a curve which is a concatenation of integral curves of the vector fields from \( \mathcal{X} \). Note that this is not quite what is implied by Chow’s theorem. Chow’s theorem gives conditions so that one can connect points with a curve whose tangent vector field lies in the subspace \( \mathcal{X}_q \) at each point. The distinction here is an admittedly subtle one, but is one worth understanding. For example, it may be harder to construct a control law connecting two points if one is only allowed to follow integral curves of the given vector fields from \( \mathcal{X} \). Also, the problem of stabilisation is one which is better adapted to the situation where one can use arbitrary curves with tangent vector fields in \( \mathcal{X} \). Therefore, in some ways the notion of possessing decoupling vector fields is not as useful as being reducible to a driftless system. In the latter case, one has greater freedom in determining the control laws available. However, the notion of decoupling vector fields is important because they can sometimes be applied in cases where the system is not reducible to a driftless system (see Exercise E4.6).

As a further note, we remark that if one wishes to avoid instantaneous velocity jumps, then one must switch between vector fields in \( \mathcal{X} \) at zero velocity. Therefore, if one wishes to use a control law designed for the system
\[
c(t) = \tilde{u}^d(t)X_u(c(t)),
\]
then one must alter the parameterisation of the curves so that one always starts and ends with zero velocity when flowing along the segment of an integral curve for one of the vector fields from \( \mathcal{X} \).

Now let us look at conditions which determine when a given vector field is a decoupling vector field.
and this shows that \( X \) is a decoupling vector field.

Now suppose that \( X \) is a decoupling vector field and let \( q \in Q \). Let \( t \mapsto c(t) \) be the integral curve of \( X \) through \( q \) and suppose that \( c(0) = q \). Define two reparameterisations, \( \tau_1 \) and \( \tau_2 \), of \( c \) with the following properties:

1. \( \tau_1(0) = 0, \tau_1'(0) = 1, \) and \( \tau_1''(0) = 0; \)
2. \( \tau_2(0) = 0, \tau_2'(0) = 1, \) and \( \tau_2''(0) = 1. \)

Let \( c_1 = c \circ \tau_1 \) and \( c_2 = c \circ \tau_2 \). We then have

\[
\nabla_{c(t)}c(t) = \nabla_{c(\tau_1(t))}c'\tau_1(t) = (\tau_1'(t))^2 \nabla_{c(\tau_1(t))}c'\tau_1(t) + \tau_1''(t)c'(\tau_1(t)).
\]

Evaluating this at \( t = 0 \) gives

\[
\nabla_{c_1(0)}c_1'(0) = \nabla_{c(\tau_1(0))}c'(\tau_1(0)) = \nabla_X X(q).
\]

Similarly for \( c_2 \) we have

\[
\nabla_{c_2(0)}c_2'(0) = \nabla_{X}(q) + X(q).
\]

Since we are assuming that \( X \) is a decoupling vector field for \( \Sigma_{aff} \) we then have

\[
\nabla_X X(q) = u_1^2 Y_\alpha(q), \quad \nabla_X X(q) + X(q) = u_2^2 Y_\alpha(q)
\]

for some \( u_1, u_2 \in \mathbb{R}^m \). This then clearly implies that \( \nabla_X X(q) \), \( X(q) \in Y_q \), as in the statement of the proposition.

As with systems reducible to a driftless system, it is possible to relate the control laws used to move the system along a decoupling vector field to control laws for the affine connection control system.

**4.5.9 Proposition** Let \( X \) be a decoupling vector field for the affine connection control system \( \Sigma_{aff} = (Q, \nabla, \mathcal{Y}) \), let \( t \mapsto c(t) \) be an integral curve for \( X \) and let \( t \mapsto \tau(t) \) be a reparameterisation for \( c \). If \( t \mapsto u(t) \in \mathbb{R}^m \) is defined by

\[
u^t(t)Y_\alpha(c \circ \tau(t)) = (\tau'(t))^2 \nabla_X(c \circ \tau(t)) + \tau''(t)X(c \circ \tau(t))\]

then \( (c \circ \tau, u) \) is a controlled trajectory for \( \Sigma_{aff} \).

**Proof** Since \( c \) is an integral curve for \( X \), \( c'(t) = X(c(t)) \). We therefore have

\[
\nabla_{(c \circ \tau)(t)} c'(t) = (\tau'(t))^2 \nabla_{c'(\tau(t))} c'(\tau(t)) + \tau''(t)c'(\tau(t)) = (\tau'(t))^2 \nabla_X(c \circ \tau(t)) + \tau''(t)X(c \circ \tau(t)).
\]

The result now follows since \( X \) is a decoupling vector field so that

\[
(\tau'(t))^2 \nabla_X X(c \circ \tau(t)) + \tau''(t)X(c \circ \tau(t)) \in Y_{c \circ \tau(t)}.
\]

Let us summarise how to deal with affine connection control systems which are kinematically controllable.
This exercise is a continuation of Exercises E2.4, E2.29, and E3.2. For the equilibria you determined for this example, determine the linearised control system, and check whether it is controllable.

This exercise is a continuation of Exercises E2.5, E2.30, and E3.3. For the equilibria you determined for this example, determine the linearised control system, and check whether it is controllable.

This exercise is a continuation of Exercises E2.6, E2.31, and E3.4. For the equilibria you determined for this example, determine the linearised control system, and check whether it is controllable.

Show that condition (ii) of Theorem 4.5.5 is equivalent to the statement that \( (Y_a : Y_b) (q) \in Y_a \) for every pair of vector fields \( Y_a, Y_b \in \mathcal{Y} \).

This exercise is a continuation of Exercises E2.7, E2.22, and E2.32. Let us use the coordinates \((r, \psi, \theta)\) as indicated in Figure E4.1. The system is a simple mechanical system with zero potential energy, and so can be represented as an affine connection control system.

(a) In the stated set of coordinates, write the vector fields \( \mathcal{Y} \) and from Exercise E2.22, write the Christoffel symbols.

(b) Use the results of your previous answer to write the control equations in affine connection control system form.

Verify that the system is reducible to a kinematic system.

(c) Use the results of your previous answer to write the control equations in affine connection control system form.

(d) Verify that the system is not reducible to a kinematic system.

(e) Show that the two vector fields

\[
X_1 = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y},
\]

\[
X_2 = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} - \frac{mh}{J} \frac{\partial}{\partial \theta}
\]

are decoupling vector fields for the system.

(f) Show that the system is kinematically controllable.

Define points \((x_1, y_1, \theta_1), (x_2, y_2, \theta_2) \in Q\) by letting \((x_i, y_i, \theta_i)\) be the point on the integral curve \(t \mapsto (x(t), y(t), \theta(t))\) for \(X_i\) which satisfies \((x(0), y(0), \theta(0)) = (0, 0, 0)\) and \((x(1), y(1), \theta(1)) = (x_i, y_i, \theta_i)\), \(i = 1, 2\). That is to say, solve the differential equation for the vector field \(X_i\), \(i = 1, 2\), with initial configuration \((0, 0, 0)\), and let \((x_i, y_i, \theta_i)\), \(i = 1, 2\), be the solution evaluated at \(t = 1\).

(g) Construct control laws which start at \((x(0), y(0), \theta(0))\) at rest, and steer the system to the point \((x_i, y_i, \theta_i), i = 1, 2\), also at rest. (That is, construct two control laws, each steering to one of the desired points.)