Exercise E2.9: Prove Proposition 2.42.

Solution: Note that

\[ s^{ij} = \begin{cases} e^i \otimes e^j, & i = j, \\ e^i \otimes e^j + e^j \otimes e^i, & i \neq j \end{cases} \]

by definition of \( e^i \otimes e^j \) and by the given definition of \( s^{ij} \). Let us prove that the set \( \{ s^{ij} \mid i, j \in \{1, \ldots, n\}, i \leq j \} \) is linearly independent. Suppose that we have a zero linear combination of these vectors:

\[ \sum_{i,j \in \{1, \ldots, n\} \atop i \leq j} c_{ij} s^{ij} = 0. \]

Applying the preceding element of \( \Sigma_2(V) \) to \((e_k, e_k)\) gives \( c_{kk} = 0 \). Applying it to \((e_k, e_l)\) for \( k < l \) gives

\[ \sum_{i,j \in \{1, \ldots, n\} \atop i \leq j} c_{ij} s^{ij}(e_k, e_l) = \sum_{i,j \in \{1, \ldots, n\} \atop i \leq j} c_{ij}(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j). \]

Note that \( k < l \) and that the sum is over \( i \leq j \). Thus the only term remaining in the sum

\[ \sum_{i,j \in \{1, \ldots, n\} \atop i \leq j} c_{ij} \delta_k^i \delta_l^j \]

is that when \( i = k \) and \( j = l \), where as the sum

\[ \sum_{i,j \in \{1, \ldots, n\} \atop i \leq j} c_{ij} \delta_l^i \delta_k^j \]

is always zero since we can never have \( i = l \) and \( j = k \) because

1. if \( i = l \) then we have \( i \leq j = k < l \) which is a contradiction and
2. if \( j = k \) we have \( j \geq i = l > k \) which is a contradiction.

Thus we have \( c_{kl} = 0 \) for \( k < l \). Thus we have \( c_{ij} = 0 \) for all \( i, j \in \{1, \ldots, n\} \) with \( i \leq j \). This gives linear independence.

Now let \( B \in \Sigma_2(V) \). Since \( B \in T_2^0(V) \) we have

\[ B = B_{ij} e^i \otimes e^j \]

where \( B_{ij} = B(e_i, e_j) \). Since \( B \) is symmetric we have \( B_{ij} = B_{ji} \) and so we can write

\[ B = B_{ij} e^i \otimes e^j = \sum_{i=1}^{n} B_{ii} e^i \otimes e^i + \sum_{i,j \in \{1, \ldots, n\} \atop i < j} B_{ij} (e^i \otimes e^j + e^j \otimes e^i) = \sum_{i,j \in \{1, \ldots, n\} \atop i \leq j} B_{ij} s^{ij}, \]

showing that \( \{ s^{ij} \mid i, j \in \{1, \ldots, n\}, i \leq j \} \) spans \( \Sigma_2(V) \), as desired. \( \blacksquare \)
Exercise E4.5: Consider a pendulum swinging atop a cart constrained to move in a line (Figure 1).

(a) What is the configuration manifold of the system, as a submanifold of \( Q_{\text{free}} \)?

(b) What is a simple abstract model for the configuration manifold?

(c) Find coordinates for the system, and indicate how they are related to the configuration of the system.

(d) Write coordinate expressions for the body and spatial translational and rotational velocities, using the notation of Section 4.1.5.

(e) Write explicit expressions for the maps \( \iota_{\text{body}}|Q \) and \( \iota_{\text{spatial}}|Q \).

Solution: (a) The free configuration space is \( Q_{\text{free}} = SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \), assuming that the pendulum consists of a massless arm, and a point mass at the end. Other assumptions for this are possible (and more realistic, although in some sense equivalent). We choose frames as follows:

1. \( O_{\text{spatial}}, O_{1,\text{body}}, \) and \( O_{2,\text{body}} \) are in the plane of motion of the pendulum/cart;
2. \( O_{1,\text{body}} \) is at the center of mass of the cart;
3. \( O_{2,\text{body}} \) is at the position of the pendulum mass;
4. \( s_3, b_{1,3}, \) and \( b_{2,3} \) are orthogonal to the plane of motion (note that we don’t really need a frame at \( O_{2,\text{body}} \) since “body” 2 is a particle);
5. \( s_1 \) points in the direction of motion of the linear motion of the cart;
6. \( b_{1,1} \) is aligned with \( s_1 \).

These assumptions are illustrated in Figure 2. With respect to these choices, an admissible point in \( Q_{\text{free}} \) has the form

\[
(I_3, (x, h, 0)), (x, h, 0) + \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} s_1,
\]

where \( \theta \in \mathbb{R} \).

(b) The configuration is essentially described by

\[
(x, \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}) \in \mathbb{R} \times SO(2).
\]

We identify \( SO(2) \) with \( S^1 \), and so take \( Q = \mathbb{R} \times S^1 \).
Figure 2. Frames for the pendulum/cart system

(c) The coordinates for the cylinder as chosen above (but with the order reversed) are okay. Thus take
\[ U = \mathbb{R} \times \mathbb{S}^1 \setminus \{(x, (u, v)) \mid u = -1, v = 0\}, \quad \phi(x, (u, v)) = (x, \text{atan}(u, v)). \]

(d) We have the spatial angular velocities as
\[ \omega_1 = (0, 0, 0), \quad \omega_2 = (0, 0, \dot{\theta}) \]
and the body angular velocities as
\[ \Omega_1 = (0, 0, 0), \quad \Omega_2 = (0, 0, \dot{\theta}). \]
The spatial and body translational velocity are both given by
\[ v_1 = (\dot{x}, 0, 0), \quad v_2 = (\dot{x} + \ell \cos \theta \dot{\theta}, \ell \sin \theta \dot{\theta}, 0). \]

(e) The maps \( \iota_{\text{body}} \) and \( \iota_{\text{spatial}} \) are obtained by concatenating the above expressions:
\[ \iota_{\text{spatial}}(x, \theta, \dot{x}, \dot{\theta}) = (\omega_1, v_1, \omega_2, v_2) \]
and
\[ \iota_{\text{body}}(x, \theta, \dot{x}, \dot{\theta}) = (\Omega_1, v_1, \Omega_2, v_2). \]

Exercise E4.7: Consider the two-axis gyroscope of Figure 3.

(a) What is the configuration manifold for the system, as a submanifold of \( Q_{\text{free}} \)?
(b) What is a simple abstract model for the configuration manifold?
(c) Find coordinates for the system, and indicate how they are related to the configuration of the system.
(d) Write coordinate expressions for the body and spatial translational and rotational velocities, using the notation of Section 4.1.5.
(e) Write explicit expressions for the maps \( \iota_{\text{body}} \mid Q \) and \( \iota_{\text{spatial}} \mid Q \).

Solution: (a) To map configurations of the system onto points in \( Q \) we choose frames \( \{O_{\text{spatial}} \mid \{s_1, s_2, s_3\} \}, \{O_{1,\text{body}} \mid \{b_{1,1}, b_{1,2}, b_{1,3}\} \}, \{O_{2,\text{body}} \mid \{b_{2,1}, b_{2,2}, b_{2,3}\} \}, \) and \( \{O_{3,\text{body}} \mid \{b_{3,1}, b_{3,2}, b_{3,3}\} \} \) as in Figure 4. Thus
1. \( O_{\text{spatial}} \) and \( O_{1,\text{body}} \) are located at the base of the gyro frame,
2. $O_{2,\text{body}}$ and $O_{3,\text{body}}$ is located at the center of rotation of the gyro,
3. $s_3$ and $b_{1,3}$ are aligned with the vertical axis of the gyro frame,
4. $b_{2,3} = b_{3,3}$ is orthogonal to the instantaneous plane of rotation of the gyro,
5. $b_{1,2} = b_{2,1}$ is aligned with the axis of rotation of the gimbal holding the gyro,
6. and the frame $\{b_{3,1}, b_{3,2}\}$ rotates with the gyro.

Next we need to write a typical point in
\[ Q_{\text{free}} = (\text{SO}(3) \times \mathbb{R}^3) \times (\text{SO}(3) \times \mathbb{R}^3) \times (\text{SO}(3) \times \mathbb{R}^3). \]

It is best to be systematic about this. The vectors $r_1, r_2, r_3 \in \mathbb{R}^3$ specifying the positions of the origins of the body frames are simple; they are both constant and of the form $r_1 = (0, 0, h_1)$, $r_2 = r_3 = (0, 0, h_2)$, for some $h_1, h_2 > 0$. The orientation of the frame $\{b_{1,1}, b_{1,2}, b_{1,3}\}$ relative to the frame $\{s_1, s_2, s_3\}$ is specified by the matrix
\[
R_1 = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} \in \text{SO}(3),
\]
for some $\theta \in \mathbb{R}$. The best way to imagine constructing $R_1$ is by noting that, since the $i$th column of the matrix is formed from the components of the vector $b_i$ with respect to the basis $\{s_1, s_2, s_3\}$, we have
\[
R_1 = \begin{bmatrix}
\langle b_{1,1}, s_1 \rangle_{\mathbb{R}^3} & \langle b_{1,2}, s_1 \rangle_{\mathbb{R}^3} & \langle b_{1,3}, s_1 \rangle_{\mathbb{R}^3} \\
\langle b_{1,1}, s_2 \rangle_{\mathbb{R}^3} & \langle b_{1,2}, s_2 \rangle_{\mathbb{R}^3} & \langle b_{1,3}, s_2 \rangle_{\mathbb{R}^3} \\
\langle b_{1,1}, s_3 \rangle_{\mathbb{R}^3} & \langle b_{1,2}, s_3 \rangle_{\mathbb{R}^3} & \langle b_{1,3}, s_3 \rangle_{\mathbb{R}^3}
\end{bmatrix}.
\]

The frame $\{b_{2,1}, b_{2,2}, b_{2,3}\}$ is oriented relative to the frame $\{b_{1,1}, b_{1,2}, b_{1,3}\}$ by the matrix
\[
R_{1 \rightarrow 2} = \begin{bmatrix}
0 & \sin \phi & \cos \phi \\
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi
\end{bmatrix} \in \text{SO}(3),
\]
for some $\phi \in \mathbb{R}$. Again, the way to think of this matrix is as

$$
R_{1\rightarrow 2} = \begin{bmatrix}
\langle \langle b_{2,1}, b_{1,1} \rangle \rangle_{\mathbb{R}^3} & \langle \langle b_{2,2}, b_{1,1} \rangle \rangle_{\mathbb{R}^3} & \langle \langle b_{2,3}, b_{1,1} \rangle \rangle_{\mathbb{R}^3} \\
\langle \langle b_{2,1}, b_{1,2} \rangle \rangle_{\mathbb{R}^3} & \langle \langle b_{2,2}, b_{1,2} \rangle \rangle_{\mathbb{R}^3} & \langle \langle b_{2,3}, b_{1,2} \rangle \rangle_{\mathbb{R}^3} \\
\langle \langle b_{2,1}, b_{1,3} \rangle \rangle_{\mathbb{R}^3} & \langle \langle b_{2,2}, b_{1,3} \rangle \rangle_{\mathbb{R}^3} & \langle \langle b_{2,3}, b_{1,3} \rangle \rangle_{\mathbb{R}^3}
\end{bmatrix}.
$$

The frame $\{b_{3,1}, b_{3,2}, b_{3,3}\}$ is related to the frame $\{b_{2,1}, b_{2,2}, b_{2,3}\}$ by the matrix

$$
R_{2\rightarrow 3} = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix} \in SO(3),
$$

for some $\psi \in \mathbb{R}$. Again, one should think of this matrix as being formed from inner products, just as illustrated above. Now we need to construct $R_2$ and $R_3$, the orientations relative to the spatial frame. For $R_2$, we need to find the components of the vectors $\{b_{2,1}, b_{2,2}, b_{2,3}\}$ relative to $\{s_1, s_2, s_3\}$. To do this we note that

$$
\langle \langle b_{2,i}, b_{1,j} \rangle \rangle_{\mathbb{R}^3} = \left\langle \left\langle b_{2,i}, \sum_{k=1}^{3} \langle b_{1,j}, s_k \rangle_{\mathbb{R}^3} s_k \right\rangle_{\mathbb{R}^3} \right\rangle_{\mathbb{R}^3} = \sum_{k=1}^{3} \langle b_{2,i}, s_k \rangle_{\mathbb{R}^3} \langle b_{1,j}, s_k \rangle_{\mathbb{R}^3}.
$$

Therefore,

$$
(R_{1\rightarrow 2})_{ij} = \sum_{k=1}^{3} (R_2)_{ki} (R_1)_{kj} \quad \Rightarrow \quad R_{1\rightarrow 2}^T = R_2^T R_1.
$$
Therefore, \( R_2 = R_1 R_{1 \rightarrow 2} \). In like fashion,

\[
R_3 = R_2 R_{2 \rightarrow 3} = R_1 R_{1 \rightarrow 2} R_{2 \rightarrow 3}.
\]

Doing the algebra gives

\[
R_2 = \begin{bmatrix}
-\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi \\
\cos \theta & \sin \theta \sin \phi & \sin \theta \cos \phi \\
0 & \cos \phi & -\sin \phi
\end{bmatrix},
\]

\[
R_3 = \begin{bmatrix}
\cos \theta \sin \phi \sin \psi - \sin \theta \cos \psi & \cos \theta \sin \phi \cos \psi + \sin \theta \sin \psi & \cos \theta \cos \phi \\
\cos \theta \cos \psi + \sin \theta \sin \phi \sin \psi & -\cos \theta \sin \psi + \sin \theta \sin \phi \cos \psi & \sin \theta \cos \phi \\
\cos \phi \sin \psi & \cos \phi \cos \psi & -\sin \phi
\end{bmatrix}.
\]

With \( R_1, R_2, R_3 \in \text{SO}(3) \) defined as above, an admissible configuration for the gyro has the form

\[
((R_1, (0, 0, h_1)), (R_2, (0, 0, h_2)), (R_3, (0, 0, h_2))) \in \mathcal{Q}_{\text{free}}.
\]

(b) Since the matrices \( R_1, R_2, \) and \( R_3 \) are formed by products of the three matrices \( R_1, R_{1 \rightarrow 2}, \) and \( R_{2 \rightarrow 3}, \) these latter matrices are really responsible for the specification of a position of the gyro. However, these matrices are each really specified by matrices in \( \text{SO}(2) \) by the correspondence

\[
R_1 \sim \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix},
\]

\[
R_{1 \rightarrow 2} \sim \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix},
\]

\[
R_{2 \rightarrow 3} \sim \begin{bmatrix}
\cos \psi & -\sin \psi \\
\sin \psi & \cos \psi
\end{bmatrix}.
\]

Thus \( Q = \text{SO}(2) \times \text{SO}(2) \times \text{SO}(2), \) or, identifying \( \text{SO}(2) \) with \( S^1, \) we take \( Q = S^1 \times S^1 \times S^1 = T^3. \)

(c) We simply define three angle coordinates for \( Q. \) Define

\[
S_1 = \left\{ ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in Q \mid x_1 = -1, y_1 = 0 \right\},
\]

\[
S_2 = \left\{ ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in Q \mid x_2 = -1, y_2 = 0 \right\},
\]

\[
S_3 = \left\{ ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in Q \mid x_3 = -1, y_3 = 0 \right\}.
\]

Then we take as a coordinate chart \((\mathcal{U}, \chi)\) where

\[
\mathcal{U} = Q \setminus (S_1 \cup S_2 \cup S_3),
\]

\[
\chi(\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}) = (\text{atan}(x_1, y_1), \text{atan}(x_2, y_2), \text{atan}(x_3, y_3)).
\]

Let us write the corresponding coordinates as \((\theta, \phi, \psi), \) and declare that \( \theta \) measures the rotational angle of the disk about the axis normal to its surface, \( \phi \) measures the angle of the base of the gyro, and \( \psi \) measures the rotation about the spindle orthogonal to the surface of the disk (see Figure 5).

(d) The computations here we omit, and just produce the answers. “1” corresponds to the base, “2” corresponds to the gimbal, and “3” corresponds to the rotor.

\[
\omega_1 = (0, 0, \dot{\theta}),
\]

\[
\omega_2 = (- \sin \theta \dot{\phi}, \cos \theta \dot{\phi}, \dot{\theta}),
\]

\[
\omega_3 = (- \sin \theta \dot{\phi} + \cos \phi \cos \theta \dot{\psi}, \cos \theta \dot{\phi} + \cos \phi \sin \theta \dot{\psi}, \dot{\theta} - \sin \phi \dot{\psi}).
\]
and
\[
\begin{align*}
\Omega_1 &= (0, 0, \dot{\theta}), \\
\Omega_2 &= (\dot{\phi}, \cos \phi \dot{\theta}, -\sin \phi \dot{\theta}), \\
\Omega_3 &= (\cos \psi \phi + \cos \phi \sin \psi \dot{\theta}, -\sin \psi \phi + \cos \phi \cos \phi \dot{\theta}, \ddot{\psi} - \sin \phi \dot{\theta}).
\end{align*}
\]

Translational velocities are, of course, zero.

(e) We concatenate:
\[
\iota_{\text{spatial}}(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = (\omega_1, v_1, \omega_2, v_2).
\]

and
\[
\iota_{\text{body}}(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = (\Omega_1, v_1, \Omega_2, v_2, \Omega_3, v_3).
\]

**Exercise E4.14:** Let $\gamma: I \to Q$ be a $C^2$-curve. Explain why the object represented in coordinates by $t \mapsto (\ddot{q}^1(t), \ldots, \ddot{q}^n(t))$ does not define a vector field along $\gamma$.

**Solution:** The quantity $(\ddot{q}^1(t), \ldots, \ddot{q}^n(t))$ does not transform like a vector field should transform. Indeed,
\[
\begin{align*}
\dot{q}^i &= \frac{\partial \ddot{q}^i}{\partial \dot{q}^j} \dot{q}^j, \\
\implies \dddot{q}^i &= \frac{\partial^2 \ddot{q}^i}{\partial q^j \partial q^k} \ddot{q}^j \ddot{q}^k + \frac{\partial \ddot{q}^i}{\partial q^j} \ddot{q}^j.
\end{align*}
\]

This is not the transformation rule for a vector field.