This test is 120 MINUTES in length and consists of 5 questions.

Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

No textbook, lecture note, calculator, computer, or other aid, is allowed.

If you need more room, there are blank pages at the end of the test. If you use these pages, you must provide clear directions to the marker, e.g. “Continued on page 10”.

Good luck!
Problem 1

Consider the system of linear equations given by:

\[
\begin{align*}
2x_1 - x_2 + x_3 &= 5, \\
3x_1 + 5x_2 - 3x_3 &= 7, \\
x_1 + 19x_2 - 13x_3 &= 1,
\end{align*}
\]

where we wish to solve for the triple \((x_1, x_2, x_3)\) of real numbers.

(a) Write the augmented matrix for this system. \[5 \text{ pts}\]

(b) Transform the augmented matrix to row-echelon form using a sequence of elementary row operations (clearly indicate which operation you perform at each step). \[5 \text{ pts}\]

(c) Find the set of all solutions of the system of linear equations by applying back-substitution to the system resulting from part (b). \[5 \text{ pts}\]

SOLUTIONS:

(a) The augmented matrix for this system is given by

\[
\begin{pmatrix}
2 & -1 & 1 & | & 5 \\
3 & 5 & -3 & | & 7 \\
1 & 19 & -13 & | & 1
\end{pmatrix}.
\]

(b) We now put this augmented matrix into row-echelon form using a sequence of elementary row operations. To begin with, we exchange rows 1 and 3 (which we denote by \(R_1 \leftrightarrow R_3\)), yielding:

\[
\begin{pmatrix}
1 & 19 & -13 & | & 1 \\
3 & 5 & -3 & | & 7 \\
2 & -1 & 1 & | & 5
\end{pmatrix}.
\]

This will make it easier to “zero out” the entries in the first column below the first row. Indeed, we can now add -3 times row 1 to row 2 (denoted by \(-3R_1 + R_2 \rightarrow R_2\)), obtaining:

\[
\begin{pmatrix}
1 & 19 & -13 & | & 1 \\
0 & -52 & 36 & | & 4 \\
2 & -1 & 1 & | & 5
\end{pmatrix}.
\]

We can similarly add -2 times row 1 to row 3 (denoted \(-2R_1 + R_3 \rightarrow R_3\)), obtaining:

\[
\begin{pmatrix}
1 & 19 & -13 & | & 1 \\
0 & -52 & 36 & | & 4 \\
0 & -39 & 27 & | & 3
\end{pmatrix}.
\]
Note that the entries on the second row are all multiples of 4; we can then multiply the second row by $\frac{1}{4}$ (denoted $\frac{1}{4}R_2 \rightarrow R_2$), yielding:

$$
\begin{pmatrix}
1 & 19 & -13 & | & 1 \\
0 & -13 & 9 & | & 1 \\
0 & -39 & 27 & | & 3
\end{pmatrix}.
$$

Similarly, the entries on the third row are all multiples of 3; we can then multiply the third row by $\frac{1}{3}$ (denoted $\frac{1}{3}R_3 \rightarrow R_3$), yielding:

$$
\begin{pmatrix}
1 & 19 & -13 & | & 1 \\
0 & -13 & 9 & | & 1 \\
0 & -13 & 9 & | & 1
\end{pmatrix}.
$$

Adding $-1$ times row 2 to row 3 (denoted $-R_2 + R_3 \rightarrow R_3$), we obtain:

$$
\begin{pmatrix}
1 & 19 & -13 & | & 1 \\
0 & -13 & 9 & | & 1 \\
0 & 0 & 0 & | & 0
\end{pmatrix},
$$

which is in row-echelon form.

(c) The system of linear equations corresponding to the matrix in row-echelon form obtained in (b) is given by:

$$
\begin{align*}
x_1 + 19x_2 - 13x_3 &= 1 \\
-13x_2 + 9x_3 &= 1.
\end{align*}
$$

Note that the entries on the last row of the matrix are all zero, and hence that row can be ignored. We use the last equation to solve for $x_2$ in terms of $x_3$, obtaining:

$$
x_2 = \frac{9}{13}x_3 - \frac{1}{13}.
$$

Substituting this value of $x_2$ in the first equation yields:

$$
x_1 = -19(\frac{9}{13}x_3 - \frac{1}{13}) + 13x_3 + 1 = -\frac{2}{13}x_3 + \frac{32}{13}.
$$

Hence, the set $S$ of solutions of our original system of linear equations is given by

$$
S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = -\frac{2}{13}x_3 + \frac{32}{13}, x_2 = \frac{9}{13}x_3 - \frac{1}{13}\}.
$$
Problem 2
Consider the real vector space \((\mathbb{R}^3, +, \cdot)\), and let \(W\) be the subset of \(\mathbb{R}^3\) consisting of all triples \((x, y, z)\) of real numbers for which \(x + 2y - z = 0\), i.e.

\[
W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y - z = 0\}.
\]

\(W\) is a vector subspace of \(\mathbb{R}^3\) and hence is itself a real vector space.

(a) **Find a basis for \(W\).**  [8 pts]

(b) **Using part (a), determine the dimension of \(W\).**  [2 pts]

**SOLUTIONS:**

(a) We have, \(\forall (x, y, z) \in \mathbb{R}^3:\)

\[
(x, y, z) \in W \iff x + 2y - z = 0
\]

\[
\iff z = x + 2y
\]

\[
\iff (x, y, z) = (x, y, x + 2y) = (x, 0, x) + (0, y, 2y) = x(1, 0, 1) + y(0, 1, 2).
\]

Let \(v_1 = (1, 0, 1), v_2 = (0, 1, 2)\). It is clear that \(v_1, v_2 \in W\). Furthermore, we have just shown that

\[
(x, y, z) \in W \iff (x, y, z) = xv_1 + yv_2.
\]

This shows that \(\{v_1, v_2\}\) is a generating set for \(W\). To show that \((v_1, v_2)\) forms a basis for \(W\), we only need to show that \(v_1, v_2\) are linearly independent. Let then \(\alpha_1, \alpha_2 \in \mathbb{R}\) such that \(\alpha_1v_1 + \alpha_2v_2 = (0, 0, 0)\). This last equation is equivalent to \(\alpha_1(1, 0, 1) + \alpha_2(0, 1, 2) = (0, 0, 0)\), which is equivalent to \((\alpha_1, \alpha_2, \alpha_1 + 2\alpha_2) = (0, 0, 0)\), which is equivalent to the system of linear equations

\[
\begin{align*}
\alpha_1 &= 0, \\
\alpha_2 &= 0, \\
\alpha_1 + 2\alpha_2 &= 0,
\end{align*}
\]

which implies \(\alpha_1 = \alpha_2 = 0\). This shows that \(v_1, v_2\) are linearly independent; since we already showed that they also form a generating set for \(W\), we can conclude that \((v_1, v_2)\) is a basis for \(W\).

(b) Since the basis for \(W\) found in (a) had exactly 2 elements, we conclude that \(W\) has dimension 2.
Problem 3
Consider the following real vector space \((W_2, +, \cdot)\) with
\[
W_2 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x > 0, y > 0\}
\]
under the following addition and scalar multiplication operations:

- **Addition:** For any \((x_1, y_1)\) and \((x_2, y_2)\) in \(W_2\),
  \[
  (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2).
  \]

- **Scalar multiplication:** For any scalar \(\alpha \in \mathbb{R}\) and \((x, y)\) in \(W_2\),
  \[
  \alpha \cdot (x, y) = (x^\alpha, y^\alpha).
  \]

Let now \(S\) be the subset of \(W_2\) defined by:
\[
S = \{(x, y) \in W_2 \mid xy^5 = 1\}.
\]
\(S\) is a vector subspace of \(W_2\), and hence, is itself a real vector space.

(a) **Find a basis for \(S\).** (Hint: Recall that for any real number \(z > 0\), and any real number \(\alpha\), we have \(z^\alpha = e^{\alpha \ln z} = (e^\alpha)^{\ln z}\)) [8 pts]

(b) **Using part (a), determine the dimension of \(S\).** [2 pts]

**SOLUTIONS:**

(a) We have, \(\forall (x, y) \in \mathbb{R}^2:\)
\[
(x, y) \in S \iff xy^5 = 1
\]
\[
\iff x = y^{-5} = e^{-5 \ln y} = (e^{-5})^{\ln y}
\]
\[
\iff (x, y) = ((e^{-5})^{\ln y}, y) = ((e^{-5})^{\ln y}, e^{\ln y}) = \ln y \cdot (e^{-5}, e) = \ln y \cdot v,
\]
with \(v = (e^{-5}, e)\). It is immediately verified that \(v \in S\). We have therefore shown that \(\forall (x, y) \in \mathbb{R}^2:\)
\[
(x, y) \in S \iff (x, y) = \ln y \cdot v.
\]
This shows that \(\{v\}\) is a generating set for \(S\); since that set consists of just \(v\) and \(v\) is not the zero vector of \(S\) (which is equal to \((1, 1)\)), it follows that \(\{v\}\) is also a linearly independent subset of \(S\).
Hence \((v)\) is a basis of \(S\).

(b) The basis of \(S\) found in (a) contains exactly one element; hence \(S\) has dimension 1.
Problem 4

For the real matrix $A$ given by

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix},$$

do the following:

(a) For $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \hat{\mathbb{R}}^4$, determine $L_A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, where $L_A$ is the linear transformation generated by $A$. [5 pts]

(b) Find a basis for the kernel of $L_A$, $\ker(A)$. [5 pts]

(c) Find a basis for the range of $L_A$, $\text{Im}(A)$. [5 pts]

SOLUTIONS:

(a) We have, $\forall \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \hat{\mathbb{R}}^4,$

$$L_A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + 3z \\ y - 2z \\ y - 2z + w \end{pmatrix}.$$ [5 pts]

(b) We have:

$$\ker(A) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \hat{\mathbb{R}}^4 \mid \begin{pmatrix} x + 3z \\ y - 2z \\ y - 2z + w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \hat{\mathbb{R}}^4 \mid x + 3z = 0, y - 2z = 0, y - 2z + w = 0 \right\}$$

Hence,

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \ker(A) \iff x + 3z = 0, y - 2z = 0, y - 2z + w = 0$$

$$\iff x = -3z, y = 2z, w = 0$$

$$\iff x = -3z, y = 2z, w = 0$$

$$\iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3z \\ 2z \\ z \\ 0 \end{pmatrix} = z \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$
Let now \( \mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \). It is immediately verified that \( \mathbf{v} \in \ker(A) \). Furthermore, we have shown above that any element of \( \ker(A) \) is a scalar multiple of \( \mathbf{v} \); hence \( \{ \mathbf{v} \} \) is a generating set for \( \ker(A) \). Since that set contains only one vector, namely \( \mathbf{v} \), and \( \mathbf{v} \) is not the zero vector, we obtain that \( \{ \mathbf{v} \} \) is a linearly independent subset of \( \ker(A) \). Hence, \( \{ \mathbf{v} \} \) is a basis for \( \ker(A) \).

(c) The Image \( \text{Im}(A) \) of \( A \) is the linear span of the column vectors of \( A \). Note that the first, second, and fourth column vectors of \( A \) are linearly independent. Indeed,

\[
\alpha \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

is equivalent to

\[
\alpha = 0, \\
\beta = 0, \\
\beta + \gamma = 0,
\]

which implies \( \alpha = \beta = \gamma = 0 \). Since these three column vectors of \( A \) are elements of \( \mathbb{R}^3 \), and since \( \mathbb{R}^3 \) is 3–dimensional, it follows that these three column vectors of \( A \) form a basis of \( \mathbb{R}^3 \). The Image of \( A \) being the linear span of all the column vectors of \( A \), contains the linear span of these three column vectors, and hence is all of \( \mathbb{R}^3 \). We therefore conclude that \( \text{Im}(A) = \mathbb{R}^3 \).
Problem 5

With the real vector space \((\mathbf{W}_2, +, \cdot)\) defined as in Problem (3), consider the mappings \(L_1 : \mathbf{W}_2 \rightarrow \mathbf{W}_2\) and \(L_2 : \mathbf{W}_2 \rightarrow \mathbf{W}_2\) defined by \(L_1((x, y)) = (x^2, y^3)\) and \(L_2((x, y)) = (2x + y, y)\) for all \((x, y) \in \mathbf{W}_2\).

(a) **Determine whether the mapping \(L_1\) is linear.** \([5 \text{ pts}]\)

(b) **Determine whether the mapping \(L_2\) is linear.** \([5 \text{ pts}]\)

**SOLUTIONS:**

(a) Let \((x_1, y_1), (x_2, y_2) \in \mathbf{W}_2\). We have

\[
L_1((x_1, y_1) + (x_2, y_2)) = L_1((x_1 x_2, y_1 y_2)) = ((x_1 x_2)^3, (y_1 y_2)^3) = (x_1^2 x_2^2, y_1^3 y_2^3) = (x_1^2, y_1^3) + (x_2^2, y_2^3) = L_1((x_1, y_1)) + L_1((x_2, y_2)).
\]

Hence the first axiom of a linear mapping is satisfied. We now verify the second axiom. Let then \(\alpha \in \mathbb{R}\), let \((x, y) \in \mathbf{W}_2\). We have:

\[
L_1(\alpha \cdot (x, y)) = L_1((x^\alpha, y^\alpha)) = ((x^\alpha)^2, (y^\alpha)^3) = (x^{2\alpha}, y^{3\alpha}) = (x^2)^\alpha, (y^3)^\alpha = \alpha \cdot (x^2, y^3) = \alpha \cdot L((x, y)).
\]

Hence the second axiom is also verified. We conclude: \(L_1\) is a linear mapping.

(b) The zero vector \(0_{\mathbf{W}_2}\) of \(\mathbf{W}_2\) is given by \(0_{\mathbf{W}_2} = (1, 1)\). We have:

\[
L_2(0) = L_2((1, 1)) = (3, 1) \neq 0_{\mathbf{W}_2};
\]

Since \(L_2\) does not map the zero vector of \(\mathbf{W}_2\) to the zero vector of \(\mathbf{W}_2\), we conclude that \(L_2\) is not a linear mapping.
(Problem 5 - Cont’d)
Space for additional work. **Indicate clearly which question you are continuing if you use this space.**
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