INSTRUCTIONS

• This test is 90 MINUTES in length and consists of 4 questions.

• Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

• SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

• No textbook, lecture note, calculator, computer, or other aid, is allowed.

• Good luck!

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1. Consider the system of linear equations given by:

\[
\begin{align*}
    x_1 + 2x_3 &= 1, \\
    x_1 + 2x_2 + 8x_3 &= 9, \\
    3x_1 - x_2 + 3x_3 &= -1,
\end{align*}
\]

where we wish to solve for the triple \((x_1, x_2, x_3)\) of real numbers.

(a) Write the **augmented matrix** for this system.  \([5 \text{ pts}]\)

(b) Transform the augmented matrix to row-echelon form using a sequence of **elementary row operations** (clearly indicate which operation you perform at each step).  \([5 \text{ pts}]\)

(c) Find the set of all solutions of the system of linear equations by applying back-substitution to the system resulting from part (b).  \([5 \text{ pts}]\)

(a) The given system of linear equations consists of **3 equations** and **3 unknowns** (namely \(x_1, x_2, x_3\)). The augmented matrix for this system will therefore have **3 rows** and **3+1=4 columns**. Ordering the unknowns by increasing index (i.e. in the order \(x_1, x_2, x_3\)), the augmented matrix \(T\) is therefore given by:

\[
T = \begin{pmatrix}
    1 & 0 & 2 & | & 1 \\
    1 & 2 & 8 & | & 9 \\
    3 & -1 & 3 & | & -1
\end{pmatrix}.
\]

(b) The augmented matrix \(T\) is not in row-echelon form; we now reduce it to row-echelon form using a sequence of elementary row operations.

(i) The elementary row operation \(-R_1 + R_2 \rightarrow R_2\) (i.e. adding - row 1 to row 2) yields the augmented matrix:

\[
T = \begin{pmatrix}
    1 & 0 & 2 & | & 1 \\
    0 & 2 & 6 & | & 8 \\
    3 & -1 & 3 & | & -1
\end{pmatrix}.
\]

(ii) The elementary row operation \(-3R_1 + R_3 \rightarrow R_3\) (i.e. adding -3 times row 1 to row 3) yields the augmented matrix:

\[
T = \begin{pmatrix}
    1 & 0 & 2 & | & 1 \\
    0 & 2 & 6 & | & 8 \\
    0 & -1 & -3 & | & -4
\end{pmatrix}.
\]
(iii) The elementary row operation \( \frac{1}{2} \mathbf{R}_2 + \mathbf{R}_3 \rightarrow \mathbf{R}_3 \) (i.e. adding \( \frac{1}{2} \) times row 2 to row 3) yields the augmented matrix:

\[
\mathbf{T} = \begin{pmatrix}
    1 & 0 & 2 & | & 1 \\
    0 & 2 & 6 & | & 8 \\
    0 & 0 & 0 & | & 0
\end{pmatrix}.
\]

\( \mathbf{T} \) is now in row-echelon form.

(c) The system of linear equations corresponding to the augmented matrix in row-echelon form obtained from the last step of (b) is given, after disregarding the redundant last row in the matrix, by:

\[
\begin{align*}
    x_1 + 2x_3 &= 1, \\
    2x_2 + 6x_3 &= 8,
\end{align*}
\]

which we solve using back-substitution. Starting from the last equation and solving there for \( x_2 \) in terms of \( x_3 \), we obtain that

\[
x_2 = 4 - 3x_3.
\]

Similarly, solving for \( x_1 \) in terms of \( x_3 \) in the first equations yields:

\[
x_1 = 1 - 2x_3.
\]

We therefore conclude that our original system of linear equations has infinitely many solutions, with each solution given by

\[
(x_1, x_2, x_3) = (1 - 2x_3, 4 - 3x_3, x_3)
\]

where \( x_3 \) is any real number.
2. Answer the following questions.

(a) **State the definition of a basis for a real vector space of dimension greater than or equal to one.** [5 pts]

(b) Consider the real vector space \((W_3, +, \cdot)\) with

\[ W_3 = \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } x > 0, y > 0, z > 0\} \]

under the following addition and scalar multiplication operations:

* **Addition:** For any \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) in \(W_3\),

\[ (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 x_2, y_1 y_2, z_1 z_2). \]

* **Scalar multiplication:** For any scalar \(\alpha \in \mathbb{R}\) and \((x, y, z)\) in \(W_3\),

\[ \alpha \cdot (x, y, z) = (x^\alpha, y^\alpha, z^\alpha). \]

Recall that the zero vector \(0\) of \(W_3\) is given by \(0 = (1, 1, 1)\). Consider the following vectors in \(W_3\): \(v_1 = (e, 1, 1)\), \(v_2 = (1, e, 1)\) and \(v_3 = (1, 1, e)\) where \(e = 2.718 \cdots\) is Euler’s number. **Show that \((v_1, v_2, v_3)\) is a basis for \(W_3\).** [5 pts]

(a) Let \((V, +, \cdot)\) be a real vector space of finite dimension (greater than or equal to one) and let \(v_1, v_2, \cdots, v_p\) be vectors in \(V\), where \(p\) is a positive integer. Then the \(p\)-tuple \((v_1, v_2, \cdots, v_p)\) is said to be a **basis** of \(V\) if

(i) The set \(\{v_1, v_2, \cdots, v_p\}\) is a **generating set** for \(V\); i.e., for any vector \(v \in V\), there exist scalars \(\alpha_1, \alpha_2, \cdots, \alpha_p\) such that

\[ v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p. \]

(ii) The set \(\{v_1, v_2, \cdots, v_p\}\) is **linearly independent**; i.e., if

\[ \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_p v_p = 0 \]

for some scalars \(\beta_1, \beta_2, \cdots, \beta_p\) with \(0\) denoting the zero vector of \(V\), then

\[ \beta_1 = \beta_2 = \cdots = \beta_p = 0. \]

(b) To show that \((v_1, v_2, v_3)\) is a basis for \(W_3\), we need to show that properties (i) and (ii) above; i.e., that \(\{v_1, v_2, v_3\}\) is both a generating set for \(W_3\) and linearly independent.
(i) For any vector \( \mathbf{v} = (x, y, z) \) in \( \mathbf{W}_3 \), we need to show that there exist properly chosen scalars \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) such that \( \mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \). Thus we need to solve for \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) in
\[
(x, y, z) = \alpha_1 \cdot (e, 1, 1) + \alpha_2 \cdot (1, e, 1) + \alpha_3 \cdot (1, 1, e).
\]
Thus
\[
(x, y, z) = (e^{\alpha_1}, 1^{\alpha_1}, 1^{\alpha_1}) + (1^{\alpha_2}, e^{\alpha_2}, 1^{\alpha_2}) + (1^{\alpha_3}, 1^{\alpha_3}, e^{\alpha_3})
\]
or equivalently
\[
(x, y, z) = (e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3}).
\]
Thus
\[
\begin{cases}
  x = e^{\alpha_1} \\
  y = e^{\alpha_2} \\
  z = e^{\alpha_3}
\end{cases}
\]
Taking the natural logarithms in each of the above equations yields
\[
\begin{cases}
  \alpha_1 = \ln x \\
  \alpha_2 = \ln y \\
  \alpha_3 = \ln z
\end{cases}
\]
Hence any vector \( \mathbf{v} = (x, y, z) \) in \( \mathbf{W}_3 \) can be written as
\[
(x, y, z) = (\ln x) \cdot (e, 1, 1) + (\ln y) \cdot (1, e, 1) + (\ln z) \cdot (1, 1, e)
\]
and thus \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is a generating set for \( \mathbf{W}_3 \).

(ii) Assume that
\[
\beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_p \mathbf{v}_p = 0
\]
and let us show that \( \beta_1 = \beta_2 = \cdots = \beta_p = 0 \). The above equation yields
\[
\beta_1 \cdot (e, 1, 1) + \beta_2 \cdot (1, e, 1) + \beta_3 \cdot (1, 1, e) = (1, 1, 1).
\]
Thus
\[
(e^{\beta_1}, e^{\beta_2}, e^{\beta_3}) = (1, 1, 1).
\]
Hence
\[
\begin{cases}
  e^{\beta_1} = 1 \\
  e^{\beta_2} = 1 \\
  e^{\beta_3} = 1
\end{cases}
\]
Taking the natural logarithms in each of the above equations yields
\[
\begin{cases}
  \beta_1 = \ln(1) = 0 \\
  \beta_2 = \ln(1) = 0 \\
  \beta_3 = \ln(1) = 0
\end{cases}
\]
Thus \( \beta_1 = \beta_2 = \beta_3 = 0 \) and the set \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is linearly independent.

Therefore \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is a basis of \( \mathbf{W}_3 \).
3. Consider the real vector space \((\mathbb{R}^3, +, \cdot)\), and let \(W\) be the subset of \(\mathbb{R}^3\) consisting of all triples \((x, y, z)\) of real numbers for which \(3y - z = 0\), i.e.

\[
W = \{(x, y, z) \in \mathbb{R}^3 \mid 3y - z = 0\}.
\]

\(W\) is a vector subspace of \(\mathbb{R}^3\) and hence is itself a real vector space.

(a) **Find a basis for** \(W\). [8 pts]

(b) **Using part (a), determine the dimension of** \(W\). [2 pts]

(c) **Consider the vector** \(w = (-3, 5, 15)\) in \(W\). **Write** \(w\) **in terms of the vectors of the basis of** \(W\) **determined in part (a).** [5 pts]

(a) We have: \(\forall (x, y, z) \in \mathbb{R}^3:\)

\[
(x, y, z) \in W \iff 3y - z = 0 \\
\iff z = 3y \\
\iff (x, y, z) = (x, y, 3y) \\
\iff (x, y, z) = (x, 0, 0) + (0, y, 3y) \\
\iff (x, y, z) = x \cdot (1, 0, 0) + y \cdot (0, 1, 3).
\]

The vectors \(v_1 = (1, 0, 0)\) and \(v_2 = (0, 1, 3)\) are clearly in \(W\); furthermore, the above calculations show that every element of \(W\) is a linear combination of \(v_1, v_2\). Hence \(\{v_1, v_2\}\) is a generating set for \(W\).

Let now \(\alpha, \beta \in \mathbb{R}\) such that \(\alpha \cdot v_1 + \beta \cdot v_2 = (0, 0, 0)\). It then follows that \((\alpha, \beta, 3\beta) = (0, 0, 0)\), which implies \(\alpha = \beta = 0\). Hence \(\{v_1, v_2\}\) is a linearly independent subset of \(W\).

We have shown that \(\{v_1, v_2\}\) is a generating set for \(W\) and that it is a linearly independent subset of \(W\); it follows therefore that \((v_1, v_2)\) is a basis for \(W\).

(b) The basis \((v_1, v_2)\) of \(W\) found in (a) has 2 elements; it follows therefore that \(W\) is a vector space of dimension 2.

(c) We have

\[
w = (-3, 5, 15) \\
= (-3, 0, 0) + (0, 5, 15) \\
= -3(1, 0, 0) + 5(0, 1, 3) \\
= -3 \cdot v_1 + 5 \cdot v_2.
\]
4. Let \((V, +, \cdot)\) be a real vector space. Let \(v_1, v_2, v_3, v_4\) be four distinct elements of \(V\) and assume that the set \(\{v_1, v_2, v_3\}\) is linearly dependent. \textbf{Determine whether or not the set} \(\{v_1, v_2, v_3, v_4\}\) \textbf{is linearly dependent}. \hfill [5 pts]

Since the set \(\{v_1, v_2, v_3\}\) is linearly dependent, there exist scalars \(\alpha_1, \alpha_2\) and \(\alpha_3\) \textbf{not all zero} such that

\[
\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 = 0.
\]

Using the above equation we can directly write

\[
\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 + (0) \cdot v_4 = 0
\]

since \((0) \cdot v_4 = 0\). Noting that not all the scalars in the above equation are zero (since not all of \(\alpha_1, \alpha_2\) and \(\alpha_3\) are zero), we conclude that the set \(\{v_1, v_2, v_3, v_4\}\) is \textbf{linearly dependent}.  