QUEEN’S UNIVERSITY
APSC174 Midterm Test #1
Monday February 11, 2013

INSTRUCTIONS

• This test is 80 MINUTES in length and consists of 3 questions.

• Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

• SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

• No textbook, lecture note, calculator, computer, or other aid, is allowed.

• Good luck!

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1. Consider the real vector space \((\mathbb{R}^3, +, \cdot)\), and let \(W\) be the subset of \(\mathbb{R}^3\) defined by:

\[ W = \{ (x, y, z) \in \mathbb{R}^3 : x - y = z \text{ and } 3x + 2y = z + 1 \}; \]

**Prove whether or not \(W\) is a vector subspace of \((\mathbb{R}^3, +, \cdot)\).**

For \(W\) to be a vector subspace of \((\mathbb{R}^3, +, \cdot)\), the following three conditions have to be met:

1. The zero vector \(0\) of \(\mathbb{R}^3\), i.e. the triple \((0, 0, 0)\) should be an element of \(W\),

2. for any elements \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) of \(W\), their sum \((x_1, y_1, z_1) + (x_2, y_2, z_2)\) should also be an element of \(W\),

3. for any real number \(\alpha\) and any element \((x, y, z)\) of \(W\), their product \(\alpha \cdot (x, y, z)\) should also be an element of \(W\).

Let us now verify these conditions in order: For the zero vector \((0, 0, 0)\), we have \(0 - 0 = 0\), whereas \(3(0) + 2(0) \neq 0 + 1 = 1\), i.e. the second relation in the definition of \(W\) is not satisfied by the zero vector \((0, 0, 0)\) of \(\mathbb{R}^3\). Hence, the zero vector \((0, 0, 0)\) of \(\mathbb{R}^3\) is not an element of \(W\), i.e. \((0, 0, 0) \not\in W\), and hence, we conclude that \(W\) is not a vector subspace of \((\mathbb{R}^3, +, \cdot)\).
2. Consider again the real vector space \((\mathbb{R}^3, +, \cdot)\), as in the previous problem.

Define now the following vectors in \(\mathbb{R}^3\):
\[
\mathbf{v}_1 = (2, 1, 0), \quad \mathbf{v}_2 = (1, 2, 0), \quad \mathbf{v}_3 = (5, 1, 0).
\]

(a) Prove whether or not the vector \(\mathbf{v}_3\) is a linear combination of the vectors \(\mathbf{v}_1, \mathbf{v}_2\).

(b) Prove whether or not \(\{\mathbf{v}_2, \mathbf{v}_3\}\) is a linearly independent subset of \(\mathbb{R}^3\).

(c) Prove whether or not \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is a linearly independent subset of \(\mathbb{R}^3\).

(a) \((7\ \text{points})\) The vector \(\mathbf{v}_3\) is a linear combination of the vectors \(\mathbf{v}_1, \mathbf{v}_2\) if and only if there exist \(\alpha, \beta \in \mathbb{R}\) such that
\[
\mathbf{v}_3 = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2.
\]

We try to solve for \(\alpha, \beta\) from the above equation; if no such \(\alpha, \beta\) do exist that satisfy the above equation, then this will show that \(\mathbf{v}_3\) is not a linear combination of \(\mathbf{v}_1, \mathbf{v}_2\). If, on the other hand, such \(\alpha, \beta\) do exist, then we can express \(\mathbf{v}_3\) as a linear combination of \(\mathbf{v}_1, \mathbf{v}_2\). Now, the equation
\[
\mathbf{v}_3 = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2
\]
is equivalent to the equation
\[
(5, 1, 0) = \alpha \cdot (2, 1, 0) + \beta \cdot (1, 2, 0)
\]
which is equivalent to
\[
(5, 1, 0) = (2\alpha + \beta, \alpha + 2\beta, 0),
\]
which is equivalent to the system of linear equations
\[
\begin{cases}
2\alpha + \beta = 5, \\
\alpha + 2\beta = 1, \\
0 = 0,
\end{cases}
\]
which, after solving (by first subtracting the first equation from the second equation multiplied by 2), yields the solution
\[
\begin{cases}
\alpha = 3, \\
\beta = -1.
\end{cases}
\]

This shows that we have
\[
\mathbf{v}_3 = 3 \cdot \mathbf{v}_1 - \mathbf{v}_2,
\]
which shows that \(\mathbf{v}_3\) is indeed a linear combination of \(\mathbf{v}_1, \mathbf{v}_2\).
(b) **(7 points)** Let \( \alpha, \beta \in \mathbb{R} \) be any real numbers such that
\[
\alpha \cdot \mathbf{v}_2 + \beta \cdot \mathbf{v}_3 = \mathbf{0}_{\mathbb{R}^3};
\]
if it then follows necessarily that \( \alpha \) and \( \beta \) must be both 0, then this will show linear independence of \( \{\mathbf{v}_2, \mathbf{v}_3\} \); if, on the other, there exist \( \alpha, \beta \) with at least one of them non-zero and such that the above equation is satisfied, then this will show linear dependence of \( \{\mathbf{v}_2, \mathbf{v}_3\} \).

We begin therefore with the equation
\[
\alpha \cdot \mathbf{v}_2 + \beta \cdot \mathbf{v}_3 = \mathbf{0}_{\mathbb{R}^3};
\]
this is equivalent to
\[
\alpha \cdot (1, 2, 0) + \beta \cdot (5, 1, 0) = (0, 0, 0),
\]
which is equivalent to
\[
(\alpha + 5\beta, 2\alpha + \beta, 0) = (0, 0, 0),
\]
which is equivalent to
\[
\begin{cases}
\alpha + 5\beta = 0, \\
2\alpha + \beta = 0, \\
0 = 0,
\end{cases}
\]
and solving this system (by first subtracting twice the first equation from the second equation) yields \( \alpha = \beta = 0 \). We have therefore shown that for any \( \alpha, \beta \in \mathbb{R} \), the relation
\[
\alpha \cdot \mathbf{v}_2 + \beta \cdot \mathbf{v}_3 = \mathbf{0}_{\mathbb{R}^3},
\]
implies necessarily that \( \alpha = \beta = 0 \). This proves that \( \{\mathbf{v}_2, \mathbf{v}_3\} \) is a **linearly independent** subset of \( \mathbb{R}^3 \).

(c) **(6 points)** We showed in (a) that \( \mathbf{v}_3 \) was a linear combination of \( \mathbf{v}_1, \mathbf{v}_2 \); as a result, the subset \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is a **linearly dependent** subset of \( \mathbb{R}^3 \).
3. Let \((V, +, \cdot)\) be a real vector space, and let \(v_1, v_2 \in V\) be two elements of \(V\). Assume \(\{v_1, v_2\}\) is a linearly independent subset of \(V\). Let now \(w_1, w_2 \in V\) be defined by

\[
\begin{align*}
    w_1 &= v_1 + 2 \cdot v_2 \\
    w_2 &= 5 \cdot v_1
\end{align*}
\]

Prove that \(\{w_1, w_2\}\) is a linearly independent subset of \(V\).

Let \(\alpha, \beta \in \mathbb{R}\) be any real numbers such that

\[
\alpha \cdot w_1 + \beta \cdot w_2 = 0_V;
\]

we have to show that this necessarily implies \(\alpha = \beta = 0\). Now, the equation

\[
\alpha \cdot w_1 + \beta \cdot w_2 = 0_V
\]

is equivalent to the equation

\[
\alpha \cdot (v_1 + 2 \cdot v_2) + \beta \cdot (5 \cdot v_1) = 0_V,
\]

which is equivalent (after rearranging and factoring terms) to the equation

\[
(\alpha + 5\beta) \cdot v_1 + (2\alpha) \cdot v_2 = 0_V;
\]

We have assumed that the vectors \(v_1, v_2\) are linearly independent. This last equation therefore implies that

\[
\begin{cases}
    \alpha + 5\beta = 0, \\
    2\alpha = 0,
\end{cases}
\]

which, after solving for \(\alpha, \beta\), yields \(\alpha = \beta = 0\). To recapitulate, we have shown that for any \(\alpha, \beta \in \mathbb{R}\), the equation

\[
\alpha \cdot w_1 + \beta \cdot w_2 = 0_V
\]

implies \(\alpha = \beta = 0\). This proves that \(\{w_1, w_2\}\) is a linearly independent subset of \(V\).