INSTRUCTIONS

- This test is 90 MINUTES in length and consists of 3 questions.
- Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.
- SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.
- No textbook, lecture note, calculator, computer, or other aid, is allowed.
- Good luck!

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1. **(25 pts)** For the real matrix \( A \) given by
\[
\begin{pmatrix}
1 & -2 & 0 & 3 \\
-1 & 0 & 2 & -1 \\
0 & 1 & 0 & -2
\end{pmatrix}
\], do the following:

(a) **Describe explicitly** the **linear transformation** \( L_A \) that it defines. (**5 pts**)

(b) Describe explicitly its **kernel** \( \ker(A) \) (i.e. \( \ker(L_A) \)) and **find a basis** for \( \ker(A) \). (**9 pts**)

(c) Describe explicitly its **range** \( \text{Im}(A) \) (i.e. \( \text{Im}(L_A) \)) and **find a basis** for \( \text{Im}(A) \). (**9 pts**)

(d) Verify the rank-nullity theorem. (**2 pts**)

(a) The linear transformation \( L_A \) defined by matrix \( A \) is the mapping from \( \mathbb{R}^4 \) (since \( A \) has 4 columns) to \( \mathbb{R}^3 \) (since \( A \) has 3 rows) defined by:
\[
L_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3
\]
\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
\mapsto
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
\mapsto
L_A\left(\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}\right) = \begin{pmatrix} x - 2y + 3w \\ -x + 2z - w \\ y - 2w \end{pmatrix}
\]

(b) We have: \( \forall \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \):
\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \in \ker(L_A) \iff L_A\left(\begin{pmatrix}
x \\ y \\ z \\ w
\end{pmatrix}\right) = 0_{\mathbb{R}^3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]
\[
\iff \begin{pmatrix} x - 2y + 3w \\ -x + 2z - w \\ y - 2w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]
\[
\iff \begin{cases}
x - 2y + 3w = 0 \\
-x + 2z - w = 0 \\
y - 2w = 0
\end{cases}
\]
\[
\iff \begin{cases}
x - 2y + 3w = 0 \\
-x + 2z - w = 0 \\
y = 2w
\end{cases}
\]
Hence, \( \ker(L_A) \) is given by:

\[
\ker(L_A) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \hat{\mathbb{R}}^4 : x = z = w \text{ and } y = 2w \right\}.
\]

We now compute a basis for \( \ker(L_A) \); first, we try to find a generating set for \( \ker(L_A) \). Using our characterization of \( \ker(L_A) \), we can write: \( \forall \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \hat{\mathbb{R}}^4 \),

\[
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \ker(L_A) \iff \begin{cases} x = w \\ y = 2w \\ z = w \end{cases}
\]

\[
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} w \\ 2w \\ w \\ w \end{pmatrix} = w \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}
\]

Letting \( v_1 \in \hat{\mathbb{R}}^4 \) be defined by \( v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \), we have therefore shown that, \( \forall \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \hat{\mathbb{R}}^4 \):

\[
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \ker(L_A) \iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = w \cdot v_1.
\]

This shows that any element in \( \ker(L_A) \) is in the linear span of \( v_1 \), and, conversely, any element in the linear span of \( v_1 \) is in \( \ker(L_A) \); in other words, the linear span \( S_{\{v_1\}} \) of \( v_1 \) is equal to \( \ker(L_A) \). Hence, \( \{v_1\} \) is a generating set for \( \ker(L_A) \). Furthermore, since \( v_1 \neq 0_{\hat{\mathbb{R}}^4} \), it follows that \( \{v_1\} \) is a linearly independent subset of \( \ker(L_A) \). Hence, \( \{v_1\} \) is a basis for
ker($L_A$). (Note that since this basis has one element, it follows that ker($L_A$) has dimension 1).

(c) Recall that Im($L_A$) is the linear span of the column vectors of $A$. Let $A_1, A_2, A_3, A_4 \in \mathbb{R}^3$ be the first, second, third and fourth column vectors of $A$, respectively; i.e., we have:

$$A_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$ 

Writing that Im($L_A$) is the linear span of $A_1, A_2, A_3, A_4$, and then using the definition of linear span, we have:

$$\text{Im}(L_A) = S_{(A_1, A_2, A_3, A_4)} = \{\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 + \alpha_3 \cdot A_3 + \alpha_4 \cdot A_4 \in \mathbb{R}^3 : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\},$$

and noting that $A_2 = -\frac{1}{2} \cdot (A_1 + A_3 + A_4)$, we obtain:

$$\begin{align*}
\text{Im}(L_A) &= \left\{\alpha_1 \cdot A_1 - \frac{1}{2} \alpha_2 \cdot A_1 - \frac{1}{2} \alpha_2 \cdot A_3 - \frac{1}{2} \alpha_2 \cdot A_4 + \alpha_3 \cdot A_3 + \alpha_4 \cdot A_4 \in \mathbb{R}^3 \mid \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\right\} \\
&= \left\{(\alpha_1 - \frac{1}{2} \alpha_2) \cdot A_1 + (\alpha_3 - \frac{1}{2} \alpha_2) \cdot A_3 + (\alpha_4 - \frac{1}{2} \alpha_2) \cdot A_4 \in \mathbb{R}^3 \mid \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\right\} \\
&= \left\{\alpha \cdot A_1 + \beta \cdot A_3 + \gamma \cdot A_4 \in \mathbb{R}^3 \mid \alpha, \beta, \gamma \in \mathbb{R}\right\} \\
&= S_{(A_1, A_3, A_4)},
\end{align*}$$

which shows that $\{A_1, A_3, A_4\}$ is a generating set for Im($L_A$). Let us now prove that the subset $\{A_1, A_3, A_4\}$ of Im($L_A$) is also linearly independent. We have, $\forall \alpha, \beta, \gamma \in \mathbb{R}$:

$$\alpha \cdot A_1 + \beta \cdot A_3 + \gamma \cdot A_4 = 0_{\mathbb{R}^3} \iff \alpha \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} \alpha + 3\gamma \\ -\alpha + 2\beta - \gamma \\ -2\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \alpha + 3\gamma = 0 \text{ and } -\alpha + 2\beta - \gamma = 0 \text{ and } -2\gamma = 0$$

$$\iff \alpha = \beta = \gamma = 0.$$

This proves that $\{A_1, A_3, A_4\}$ is a linearly independent subset of Im($L_A$). Hence, $(A_1, A_3, A_4)$ is a basis for Im($L_A$). (Note that since this basis has 3 elements, it follows that Im($L_A$) has dimension 3).

(d) Since $L_A : \mathbb{R}^4 \to \mathbb{R}^3$, we have from the rank-nullity theorem that

$$\text{rank}(L_A) + \text{nullity}(L_A) = n$$
where \( n = 4 \) is the dimension of the domain vector space \( \mathbb{R}^4 \) (or the number of columns in \( A \)), \( \text{rank}(L_A) \) is the dimension of \( \text{Im}(L_A) \), and \( \text{nullity}(L_A) \) is the dimension of \( \text{ker}(L_A) \). From part (b) above, since the basis of \( \text{ker}(L_A) \) has one vector, we obtain that

\[
\text{nullity}(L_A) = 1.
\]

Similarly, from part (c), we note that the basis of \( \text{Im}(L_A) \) has three vectors and thus

\[
\text{rank}(L_A) = 3.
\]

Hence

\[
\text{rank}(L_A) + \text{nullity}(L_A) = 3 + 1 = 4 = n
\]

which verifies the rank-nullity theorem.
2. (10 pts) Let \((V, +, \cdot)\) and \((W, +, \cdot)\) be two real vector spaces, and let \(L_1 : V \to W\) and \(L_2 : V \to W\) be two linear mappings. We define the mapping \(L : V \to W\) as follows:

\[
\forall v \in V, \quad L(v) = 3 \cdot L_1(v) - L_2(v).
\]

**Determine whether or not** the mapping \(L\) is linear.

We have \(\forall v_1, v_2 \in V:\)

\[
L(v_1 + v_2) = 3 \cdot L_1(v_1 + v_2) - L_2(v_1 + v_2)
\]

\[
= 3 \cdot L_1(v_1) + 3 \cdot L_1(v_2) - L_2(v_1 + v_2)
\]

\[
= 3 \cdot L_1(v_1) + 3 \cdot L_1(v_2) - L_2(v_1) - L_2(v_2)
\]

\[
= [3 \cdot L_1(v_1) - L_2(v_1)] + [3 \cdot L_1(v_2) - L_2(v_2)]
\]

\[
= L(v_1) + L(v_2),
\]

where the second equality above follows from the linearity of \(L_1\) and the third equality follows from the linearity of \(L_2\). Furthermore, we have \(\forall \alpha \in \mathbb{R}, \forall v \in V:\)

\[
L(\alpha \cdot v) = 3 \cdot L_1(\alpha \cdot v) - L_2(\alpha \cdot v)
\]

\[
= (3\alpha) \cdot L_1(v) - L_2(\alpha \cdot v)
\]

\[
= (3\alpha) \cdot L_1(v) - \alpha \cdot L_2(v)
\]

\[
= \alpha \cdot [3 \cdot L_1(v) - L_2(v)]
\]

\[
= \alpha \cdot L(v),
\]

where again the second equality above follows from the linearity of \(L_1\) and the third equality from the linearity of \(L_2\). In summary, we have shown:

\[
\forall v_1, v_2 \in V : \quad L(v_1 + v_2) = L(v_1) + L(v_2)
\]

\[
\forall \alpha \in \mathbb{R}, \forall v \in V : \quad L(\alpha \cdot v) = \alpha \cdot L(v).
\]

Hence, \(L\) is a linear mapping.
3. (10 pts) Let \((V, +, \cdot)\) and \((W, +, \cdot)\) be real vector spaces, and let \(L : V \to W\) be a linear mapping such that
\[
\ker(L) = \{0_V\}.
\]
Let now \(v_1, v_2, v_3 \in V\), and assume that the set \(\{v_1, v_2, v_3\}\) is linearly independent. Determine whether or not the set \(\{L(v_1), L(v_2), L(v_3)\}\) is linearly independent.

For \(\alpha_1, \alpha_2, \alpha_3\) in \(\mathbb{R}\), assume that
\[
\alpha_1 \cdot L(v_1) + \alpha_2 \cdot L(v_2) + \alpha_3 \cdot L(v_3) = 0_W.
\]
Then by the linearity of \(L\), we obtain:
\[
L(\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3) = 0_W,
\]
which directly implies that
\[
\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 \in \ker(L).
\]
But since \(\ker(L) = \{0_V\}\), we must then have that
\[
\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 = 0_V.
\]
Now, the above equality together with the fact that the set \(\{v_1, v_2, v_3\}\) is linearly independent imply that we must have
\[
\alpha_1 = \alpha_2 = \alpha_3 = 0.
\]
It thus follows that the family of vectors \(L(v_1), L(v_2), L(v_3)\) is linearly independent.