QUEEN’S UNIVERSITY
APSC174 Midterm Test #1
Monday February 10, 2014

INSTRUCTIONS

• This test is 80 MINUTES in length and consists of 3 questions.

• Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

• SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

• No textbook, lecture note, calculator, computer, or other aid, is allowed.

• Good luck!

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1. Consider the real vector space \((\mathbb{R}^3, +, \cdot)\), and let \(W\) be the subset of \(\mathbb{R}^3\) defined by:

\[
W = \{ (x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0 \};
\]

Prove whether or not \(W\) is a vector subspace of \((\mathbb{R}^3, +, \cdot)\).

For \(W\) to be a vector subspace of \((\mathbb{R}^3, +, \cdot)\), the following three conditions have to be met:

1. \(0_{\mathbb{R}^3} = (0, 0, 0) \in W\),
2. \(\forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in W : (x_1, y_1, z_1) + (x_2, y_2, z_2) \in W\),
3. \(\forall \alpha \in \mathbb{R}, \forall (x, y, z) \in W : \alpha \cdot (x, y, z) \in W\).

Let us now verify these conditions in order:

- **(4 points)** We have \(0 + 2(0) - 0 = 0\), hence \((0, 0, 0) \in W\). In other words, the zero vector of \(\mathbb{R}^3\) is indeed an element of \(W\), and condition (1) for a vector subspace is verified.

- **(8 points)** Let now \((x_1, y_1, z_1), (x_2, y_2, z_2) \in W\); we have \((x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)\); we have \((x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2) = (x_1 + 2y_1 - z_1) + (x_2 + 2y_2 - z_2) = 0 + 0 = 0\), and hence \((x_1, y_1, z_1) + (x_2, y_2, z_2) \in W\). This shows that condition (2) for a vector subspace is also verified.

- **(8 points)** Let now \(\alpha \in \mathbb{R}\) and \((x, y, z) \in W\); we have \(\alpha \cdot (x, y, z) = (\alpha x, \alpha y, \alpha z)\), and \(\alpha x + 2\alpha y - \alpha z = \alpha(x + 2y - z) = 0\), which shows that \(\alpha \cdot (x, y, z) \in W\). Hence condition (3) for a vector subspace is also verified.

We conclude: \(W\) is a vector subspace of the real vector space \((\mathbb{R}^3, +, \cdot)\).
2. Consider again the real vector space \((\mathbb{R}^3,+,\cdot)\), as in the previous problem.

Define now the following vectors in \(\mathbb{R}^3\):

\[
\mathbf{v}_1 = (1, 1, 0), \quad \mathbf{v}_2 = (1, 3, 0), \quad \mathbf{v}_3 = (1, -5, 0).
\]

(a) \textbf{Prove whether or not the vector } \mathbf{v}_3 \text{ is a linear combination of the vectors } \mathbf{v}_1, \mathbf{v}_2.

(b) \textbf{Prove whether or not } \{\mathbf{v}_1\} \text{ is a linearly independent subset of } \mathbb{R}^3.

(c) \textbf{Prove whether or not } \{\mathbf{v}_2, \mathbf{v}_3\} \text{ is a linearly independent subset of } \mathbb{R}^3.

(d) \textbf{Prove whether or not } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ is a linearly independent subset of } \mathbb{R}^3.

(a) \textbf{(5 points)} The vector \(\mathbf{v}_3\) is a linear combination of the vectors \(\mathbf{v}_1, \mathbf{v}_2\) if and only if there exist \(\alpha, \beta \in \mathbb{R}\) such that

\[
\mathbf{v}_3 = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2.
\]

We try to solve for \(\alpha, \beta\) from the above equation; if no such \(\alpha, \beta\) do exist that satisfy the above equation, then this will show that \(\mathbf{v}_3\) is not a linear combination of \(\mathbf{v}_1, \mathbf{v}_2\). If, on the other hand, such \(\alpha, \beta\) do exist, then we can express \(\mathbf{v}_3\) as a linear combination of \(\mathbf{v}_1, \mathbf{v}_2\). Now, the equation

\[
\mathbf{v}_3 = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2
\]

is equivalent to the equation

\[
(1, -5, 0) = \alpha \cdot (1, 1, 0) + \beta \cdot (1, 3, 0)
\]

which is equivalent to

\[
(1, -5, 0) = (\alpha + \beta, \alpha + 3\beta, 0),
\]

which is equivalent to the system of linear equations

\[
\begin{align*}
\alpha + \beta &= 1, \\
\alpha + 3\beta &= -5,
\end{align*}
\]

which, after solving (by first subtracting the first equation from the second equation), yields the solution

\[
\begin{align*}
\alpha &= 4, \\
\beta &= -3.
\end{align*}
\]

This shows that we have

\[
\mathbf{v}_3 = 4 \cdot \mathbf{v}_1 + (-3) \cdot \mathbf{v}_2,
\]

which shows that \(\mathbf{v}_3\) \textbf{is indeed} a linear combination of \(\mathbf{v}_1, \mathbf{v}_2\).
(b) (5 points) Let $\alpha \in \mathbb{R}$ be any real number such that $\alpha \cdot \mathbf{v}_1 = \mathbf{0}_{\mathbb{R}^3}$; if it then follows that $\alpha$ must be 0, then this will show linear independence of $\{\mathbf{v}_1\}$; if on the other hand there exists $\alpha \neq 0$ such that $\alpha \cdot \mathbf{v}_1 = \mathbf{0}_{\mathbb{R}^3}$, then this will show linear dependence of $\{\mathbf{v}_1\}$. We begin therefore with the equation

$$\alpha \cdot \mathbf{v}_1 = \mathbf{0}_{\mathbb{R}^3};$$

this is equivalent to

$$\alpha \cdot (1, 1, 0) = (0, 0, 0),$$

which is equivalent to

$$(\alpha, \alpha, 0) = (0, 0, 0),$$

which is equivalent to

$$\begin{cases} 
\alpha = 0, \\
\alpha = 0, \\
0 = 0,
\end{cases}$$

which yields $\alpha = 0$. We have therefore shown that for any $\alpha \in \mathbb{R}$, the relation

$$\alpha \cdot \mathbf{v}_1 = \mathbf{0}_{\mathbb{R}^3},$$

implies necessarily that $\alpha = 0$. This proves that $\{\mathbf{v}_1\}$ is a linearly independent subset of $\mathbb{R}^3$.

(c) (5 points) Let $\alpha, \beta \in \mathbb{R}$ be any real numbers such that

$$\alpha \cdot \mathbf{v}_2 + \beta \cdot \mathbf{v}_3 = \mathbf{0}_{\mathbb{R}^3};$$

if it then follows necessarily that $\alpha$ and $\beta$ must be both 0, then this will show linear independence of $\{\mathbf{v}_2, \mathbf{v}_3\}$; if, on the other, there exist $\alpha, \beta$ with at least one of them non-zero and such that the above equation is satisfied, then this will show linear dependence of $\{\mathbf{v}_2, \mathbf{v}_3\}$. We begin therefore with the equation

$$\alpha \cdot \mathbf{v}_2 + \beta \cdot \mathbf{v}_3 = \mathbf{0}_{\mathbb{R}^3};$$

this is equivalent to

$$\alpha \cdot (1, 3, 0) + \beta \cdot (1, -5, 0) = (0, 0, 0),$$

which is equivalent to

$$(\alpha + \beta, 3\alpha - 5\beta, 0) = (0, 0, 0),$$
which is equivalent to

\[
\begin{align*}
\alpha + \beta &= 0, \\
3\alpha - 5\beta &= 0, \\
0 &= 0,
\end{align*}
\]

and solving this system (by first solving for \(\alpha\) in terms of \(\beta\) from the first equation) yields \(\alpha = \beta = 0\). We have therefore shown that for any \(\alpha, \beta \in \mathbb{R}\), the relation

\[
\alpha \cdot \mathbf{v}_2 + \beta \cdot \mathbf{v}_3 = \mathbf{0}_{\mathbb{R}^3},
\]

implies necessarily that \(\alpha = \beta = 0\). This proves that \(\{\mathbf{v}_2, \mathbf{v}_3\}\) is a \textbf{linearly independent} subset of \(\mathbb{R}^3\).

(d) \textbf{(5 points)} We showed in (a) that \(\mathbf{v}_3\) was a linear combination of \(\mathbf{v}_1, \mathbf{v}_2\); as a result, the subset \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is a \textbf{linearly dependent} subset of \(\mathbb{R}^3\).
3. Let \((V, +, \cdot)\) be a real vector space, and let \(v_1, v_2 \in V\) be two elements of \(V\). Assume \(\{v_1, v_2\}\) is a linearly independent subset of \(V\). Let now \(w_1, w_2 \in V\) be defined by
\[
\begin{align*}
  w_1 &= v_1 + 2 \cdot v_2 \\
  w_2 &= 3 \cdot v_1
\end{align*}
\]
Prove that \(\{w_1, w_2\}\) is a linearly independent subset of \(V\).

Let \(\alpha, \beta \in \mathbb{R}\) be any real numbers such that
\[
\alpha \cdot w_1 + \beta \cdot w_2 = 0_V;
\]
we have to show that this necessarily implies \(\alpha = \beta = 0\). Now, the equation
\[
\alpha \cdot w_1 + \beta \cdot w_2 = 0_V
\]
is equivalent to the equation
\[
\alpha \cdot (v_1 + 2 \cdot v_2) + \beta \cdot (3 \cdot v_1) = 0_V,
\]
which is equivalent (after rearranging and factoring terms) to the equation
\[
(\alpha + 3\beta) \cdot v_1 + (2\alpha) \cdot v_2 = 0_V;
\]
We have assumed that the family \(\{v_1, v_2\}\) of vectors was linearly independent. This last equation therefore implies that
\[
\begin{aligned}
  \alpha + 3\beta &= 0, \\
  2\alpha &= 0,
\end{aligned}
\]
which, after solving for \(\alpha, \beta\), yields \(\alpha = \beta = 0\). To recapitulate, we have shown that for any \(\alpha, \beta \in \mathbb{R}\), the equation
\[
\alpha \cdot w_1 + \beta \cdot w_2 = 0_V
\]
implies \(\alpha = \beta = 0\). This proves that \(\{w_1, w_2\}\) is a linearly independent subset of \(V\).