QUEEN’S UNIVERSITY
APSC174 Midterm Test #3
Monday March 24, 2014

INSTRUCTIONS

• This test is 90 MINUTES in length and consists of 3 questions.

• Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

• SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

• No textbook, lecture note, calculator, computer, or other aid, is allowed.

• Good luck!

<table>
<thead>
<tr>
<th>Question</th>
<th>Mark Available</th>
<th>Received</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>
1. (25 pts) For the real the matrix \( A \) given by
\[
A = \begin{pmatrix}
1 & 0 & 2 & -1 \\
-1 & 1 & -2 & 1
\end{pmatrix},
\]
do the following:

(a) **Specify** the linear transformation \( L_A \) that it defines. (5 pts)
(b) Specify its kernel \( \ker(A) \) (i.e. \( \ker(L_A) \)) and **find a basis** for \( \ker(A) \). (10 pts)
(c) Specify its range \( \text{Im}(A) \) (i.e. \( \text{Im}(L_A) \)) and **find a basis** for \( \text{Im}(A) \). (10 pts)

(a) The linear transformation \( L_A \) defined by matrix \( A \) is the mapping from \( \mathbb{R}^4 \) (since \( A \) has 4 columns) to \( \mathbb{R}^2 \) (since \( A \) has 2 rows) defined by:
\[
L_A : \mathbb{R}^4 \rightarrow \mathbb{R}^2 \\
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \mapsto L_A\left(\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}\right) = \begin{pmatrix}
x + 2z - w \\
x - y - 2z + w
\end{pmatrix}
\]

(b) We have: \( \forall \begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \in \ker(L_A) \iff L_A\left(\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}\right) = 0_{\mathbb{R}^2} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
\[
\iff \begin{pmatrix}
x + 2z - w \\
x - y - 2z + w
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
\[
\iff \begin{cases}
x + 2z - w = 0 \\
x - y - 2z + w = 0
\end{cases}
\]
\[
\iff \begin{cases}
w = x + 2z \\
y = x + 2z - w
\end{cases}
\]
\[
\iff \begin{cases}
w = x + 2z \\
y = 0
\end{cases}
\]
Hence, \( \ker(L_A) \) is given by:
\[
\ker(L_A) = \left\{ \begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} \in \mathbb{R}^4 : w = x + 2z \text{ and } y = 0 \right\}.
\]
We now compute a basis for \( \ker(L_A) \); first, we try to find a generating set for \( \ker(L_A) \). Using our characterization of \( \ker(L_A) \), we can write: \( \forall \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4, \)

\[
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \ker(L_A) \iff \begin{cases} w = x + 2z \\ y = 0 \end{cases}
\]

\[
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \\ 1 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}
\]

Let \( \mathbf{v}_1 \in \ker(L_A) \) be defined by \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \), and let \( \mathbf{v}_2 \in \ker(L_A) \) be defined by \( \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \).

We have therefore shown that, \( \forall \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4: \)

\[
\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \ker(L_A) \iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x \cdot \mathbf{v}_1 + z \cdot \mathbf{v}_2.
\]

This shows that any element in \( \ker(L_A) \) is in the linear span of \( \mathbf{v}_1, \mathbf{v}_2 \), and, conversely, any element in the linear span of \( \mathbf{v}_1, \mathbf{v}_2 \) is in \( \ker(L_A) \); in other words, the linear span \( \mathcal{S}_{\{\mathbf{v}_1, \mathbf{v}_2\}} \) of \( \mathbf{v}_1, \mathbf{v}_2 \) is equal to \( \ker(L_A) \). Hence, \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is a generating set for \( \ker(L_A) \). We now verify that \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is a linearly independent subset of \( \ker(L_A) \). Let then \( \alpha_1, \alpha_2 \in \mathbb{R} \) such that \( \alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 = \mathbf{0}_{\mathbb{R}^4} \). This implies that

\[
\alpha_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 0 \\ 0 \\ \alpha_1 + 2\alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

which implies that \( \alpha_1 = \alpha_2 = 0 \). Hence, \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is a linearly independent subset of \( \ker(L_A) \). Hence, \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is a basis for \( \ker(L_A) \). (Note that since this basis has two elements, it follows that \( \ker(L_A) \) has dimension 2).
(c) Recall that $\text{Im}(L_A)$ is the linear span of the column vectors of $A$. Let $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$ be the first, second, third, and fourth column vectors of $A$, respectively; i.e., we have:

$$A_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ A_3 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \ A_4 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$ 

Writing that $\text{Im}(L_A)$ is the linear span of $A_1, A_2, A_3, A_4$, and then using the definition of linear span, we have:

$$\text{Im}(L_A) = S_{(A_1, A_2, A_3, A_4)} = \{\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 + \alpha_3 \cdot A_3 + \alpha_4 \cdot A_4 \in \mathbb{R}^2 : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\},$$

and noting that $A_3 = 2 \cdot A_1$ and $A_4 = -A_1$ we obtain:

$$\text{Im}(L_A) = \{\alpha \cdot A_1 + \beta \cdot A_2 \in \mathbb{R}^2 : \alpha, \beta \in \mathbb{R}\} = S_{(A_1, A_2)},$$

which shows that $\{A_1, A_2\}$ is a generating set for $\text{Im}(L_A)$. Let us now prove that the subset $\{A_1, A_2\}$ of $\text{Im}(L_A)$ is also linearly independent. We have, $\forall \alpha, \beta \in \mathbb{R}$:

$$\alpha \cdot A_1 + \beta \cdot A_2 = 0_{\mathbb{R}^2} \iff \alpha \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} \alpha \\ -\alpha + \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} \alpha = 0 \\ -\alpha + \beta = 0 \end{cases} \Rightarrow \alpha = \beta = 0.$$ 

This proves that $\{A_1, A_2\}$ is a linearly independent subset of $\text{Im}(L_A)$. Hence, $(A_1, A_2)$ is a basis for $\text{Im}(L_A)$. (Not that since this basis has 2 elements, it follows that $\text{Im}(L_A)$ has dimension 2).
2. (10 pts) Consider the real vector spaces \((\mathbb{R}^2, +, \cdot)\) and \((\mathbb{R}^3, +, \cdot)\), and define the mapping \(L : \mathbb{R}^2 \to \mathbb{R}^3\) by
\[
L((x, y)) = (x^2 + y, x + y^2, x + y), \quad \forall (x, y) \in \mathbb{R}^2.
\]
Determine whether or not the mapping \(L\) is linear.

To show that \(L\) is linear, we have to show that:

(i) \(\forall v_1, v_2 \in V: L(v_1 + v_2) = L(v_1) + L(v_2)\),

(ii) \(\forall \alpha \in \mathbb{R}, \forall v \in V: L(\alpha \cdot v) = \alpha \cdot L(v)\).

For \(L\) to be linear, both of the above statements have to be true; if either of them fails, then we conclude that \(L\) is not linear. Let us first check whether property (ii) is satisfied by \(L\). Let then \(\alpha \in \mathbb{R}\), and let \((x, y) \in \mathbb{R}^2\). We have:
\[
L(\alpha \cdot (x, y)) = L((\alpha x, \alpha y)) = ((\alpha x)^2 + \alpha y, \alpha x + (\alpha y)^2, \alpha x + \alpha y)
= (\alpha^2 x^2 + \alpha y, \alpha x + \alpha^2 y^2, \alpha x + \alpha y),
\]
whereas
\[
\alpha \cdot L((x, y)) = \alpha \cdot (x^2 + y, x + y^2, x + y)
= (\alpha x^2 + \alpha y, \alpha x + \alpha y^2, \alpha (x + y))
= (\alpha x^2 + \alpha y, \alpha x + \alpha y^2, \alpha x + \alpha y).
\]

Note that the entries of \(L(\alpha \cdot (x, y))\) contain terms with factors \(\alpha^2\), whereas the entries of \(\alpha \cdot L((x, y))\) do not. This suggests that for some choices of \(\alpha\) and \((x, y)\), \(L(\alpha \cdot (x, y))\) will not be equal to \(\alpha \cdot L((x, y))\). Let then \(\alpha = 2\) and \((x, y) = (1, 0)\); for these choices, we have
\[
L(2 \cdot (1, 0)) = L((2, 0)) = (4, 2, 2),
\]
whereas
\[
2 \cdot L((1, 0)) = 2 \cdot (1, 1, 1) = (2, 2, 2).
\]
Hence,
\[
L(2 \cdot (1, 0)) \neq 2 \cdot L((1, 0)).
\]
It follows that property (ii) of a linear mapping is not satisfied by \(L\) (if it were, we would have \(L(\alpha \cdot (x, y)) = \alpha \cdot L((x, y))\) for any choice of \(\alpha \in \mathbb{R}\) and any choice of \((x, y) \in \mathbb{R}^2\)). We conclude therefore that \(L\) is not linear.
3. (10 pts) Let \((V, +, \cdot)\) and \((W, +, \cdot)\) be real vector spaces, and let \(L : V \rightarrow W\) be a linear mapping. Let now \(v_1, v_2, v_3 \in V\). Assume that \(\text{Ker}(L) = \{0_V\}\) and assume also that the vectors \(v_1, v_2, v_3\) are linearly independent. Determine whether or not the vectors \(L(v_1), L(v_2), L(v_3)\) are linearly independent.

Let \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\) be any real numbers such that

\[
\alpha_1 \cdot L(v_1) + \alpha_2 \cdot L(v_2) + \alpha_3 \cdot L(v_3) = 0_w.
\]

If it then follows that \(\alpha_1 = \alpha_2 = \alpha_3 = 0\), then we conclude that the vectors \(L(v_1), L(v_2), L(v_3)\) form a linearly independent family. Now,

\[
\alpha_1 \cdot L(v_1) + \alpha_2 \cdot L(v_2) + \alpha_3 \cdot L(v_3) = 0_w
\]

is equivalent (by linearity of \(L\)) to:

\[
L(\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3) = 0_w
\]

which is equivalent to

\[
\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 \in \ker(L),
\]

and since by assumption \(\ker(L) = \{0_V\}\), the previous equality implies

\[
\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 = 0_v,
\]

and since the vectors \(v_1, v_2, v_3\) are assumed to form a linearly independent family, the previous equality implies \(\alpha_1 = \alpha_2 = \alpha_3 = 0\). To recapitulate, we have shown that for any \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\), the equality

\[
\alpha_1 \cdot L(v_1) + \alpha_2 \cdot L(v_2) + \alpha_3 \cdot L(v_3) = 0_w.
\]

implies \(\alpha_1 = \alpha_2 = \alpha_3 = 0\). We conclude that the vectors \(L(v_1), L(v_2), L(v_3)\) form a linearly independent family.