INSTRUCTIONS

• This examination is 3 hours in length and consists of 6 questions.

• READ THE QUESTIONS CAREFULLY!

• Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page and provide clear directions to the marker.

• SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

• No textbook, lecture note, calculator, computer, or other aid of any sort is allowed.

• PLEASE NOTE: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer the exam questions as written.

• Good luck!

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1. Given a real number \( a \), consider the system of linear equations given by:

\[
\begin{align*}
    x_1 + x_2 + 2x_4 &= a, \\
    x_1 - x_2 + 2x_4 &= 2, \\
    -2x_1 - x_2 + 5x_3 + x_4 &= 0, \\
    x_1 + x_2 - 2x_3 &= 2,
\end{align*}
\]

where we wish to solve for the quadruple \((x_1, x_2, x_3, x_4)\) of real numbers.

(a) Write the augmented matrix for this system. (4 pts)

(b) Transform the augmented matrix to row-echelon form using a sequence of elementary row operations (clearly indicate which operation you perform at each step). (6 pts)

(c) Using (b), determine all the values of \( a \) for which the system has no solution. (2 pts)

(d) Using (b), determine all the values of \( a \) for which the system has a solution. (2 pts)

(e) For those values of \( a \) obtained in (d) for which the system has a solution, determine the set of all solutions to the original system of linear equations. (6 pts)

(a) The augmented matrix for this system is given by:

\[
\begin{pmatrix}
1 & 1 & 0 & 2 & | & a \\
1 & -1 & 0 & 2 & | & 2 \\
-2 & -1 & 5 & 1 & | & 0 \\
1 & 1 & -2 & 0 & | & 2 \\
\end{pmatrix}
\]

(b) Exchanging rows 1 and 4 \((R1 \leftrightarrow R4)\) yields:

\[
\begin{pmatrix}
1 & 1 & -2 & 0 & | & 2 \\
1 & -1 & 0 & 2 & | & 2 \\
-2 & -1 & 5 & 1 & | & 0 \\
1 & 1 & 0 & 2 & | & a \\
\end{pmatrix}
\]

Adding \(-1\times\) row 1 to row 2 \((-R1 + R2 \rightarrow R2)\) yields

\[
\begin{pmatrix}
1 & 1 & -2 & 0 & | & 2 \\
0 & -2 & 2 & 2 & | & 0 \\
-2 & -1 & 5 & 1 & | & 0 \\
1 & 1 & 0 & 2 & | & a \\
\end{pmatrix}
\]

Adding twice row 1 to row 3 \((2R1 + R3 \rightarrow R3)\) yields

\[
\begin{pmatrix}
1 & 1 & -2 & 0 & | & 2 \\
0 & -2 & 2 & 2 & | & 0 \\
0 & 1 & 1 & 1 & | & 4 \\
1 & 1 & 0 & 2 & | & a \\
\end{pmatrix}
\]
Adding $\frac{1}{2} \times$ row 2 to row 3 ($\frac{1}{2}R2 + R3 \rightarrow R3$) yields
\[
\begin{pmatrix}
1 & 1 & -2 & 0 & | & 2 \\
0 & -2 & 2 & 2 & | & 0 \\
0 & 0 & 2 & 2 & | & 4 \\
1 & 1 & 0 & 2 & | & a \\
\end{pmatrix}
\]

Adding $-1 \times$ row 1 to row 4 ($-R1 + R4 \rightarrow R4$) yields
\[
\begin{pmatrix}
1 & 1 & -2 & 0 & | & 2 \\
0 & -2 & 2 & 2 & | & 0 \\
0 & 0 & 2 & 2 & | & 4 \\
0 & 0 & 2 & 2 & | & a - 2 \\
\end{pmatrix}
\]

Adding $-1 \times$ row 3 to row 4 ($-R3 + R4 \rightarrow R4$) yields
\[
\begin{pmatrix}
1 & 1 & -2 & 0 & | & 2 \\
0 & -2 & 2 & 2 & | & 0 \\
0 & 0 & 2 & 2 & | & 4 \\
0 & 0 & 0 & 0 & | & a - 6 \\
\end{pmatrix}
\]

The above matrix is now in row-echelon form.

(c) It can be seen from the last row of the augmented matrix in row echelon form that the given system of linear equations has no solution if and only if
\[a - 6 \neq 0,
\]
that is, if and only if
\[a \neq 6.
\]

(d) It can similarly be seen from part (c) that the system of linear equations has a solution if and only if
\[a = 6.
\]

(e) For $a = 6$, the augmented matrix in row-echelon form is given by
\[
\begin{pmatrix}
1 & 1 & -2 & 0 & | & 2 \\
0 & -2 & 2 & 2 & | & 0 \\
0 & 0 & 2 & 2 & | & 4 \\
0 & 0 & 0 & 0 & | & 0 \\
\end{pmatrix}
\]

The system of linear equation corresponding to this augmented matrix in row echelon form is given by:
\begin{align*}
x_1 + x_2 - 2x_3 &= 2 \\
-2x_2 + 2x_3 + 2x_4 &= 0 \\
2x_3 + 2x_4 &= 4
\end{align*} \tag{1}

and solving for \(x_3\) in terms of \(x_4\) in the last equation, we obtain:

\[x_3 = 2 - x_4\] \tag{2}

and substituting this value of \(x_3\) back in the second equation, and solving for \(x_2\), we obtain:

\[x_2 = 2\] \tag{3}

and substituting the above values of \(x_3\) and \(x_2\) in the first equation, and solving for \(x_1\), we obtain:

\[x_1 = 4 - 2x_4\] \tag{4}

We can therefore state: When \(a = 6\), the system of linear equations has infinitely many solutions and the set \(S\) of all its solutions is given by:

\[S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 4 - 2x_4, x_2 = 2, x_3 = 2 - x_4\}. \] \tag{5}
2. Let $L : V \to W$ be a mapping where $(V, +, \cdot)$ and $(W, +, \cdot)$ are real vector spaces.

(a) Define what it means for the mapping $L$ to be linear. (2 pts)

(b) Show that if $L$ is linear, then $L(a v_1 + b v_2) = a L(v_1) + b (L v_2)$ for all scalars $a, b \in \mathbb{R}$ and vectors $v_1, v_2 \in V$. (4 pts)

(c) Show that if $L(a v_1 + b v_2) = a L(v_1) + b (L v_2)$ for all scalars $a, b \in \mathbb{R}$ and vectors $v_1, v_2 \in V$, then $L$ is linear. (4 pts)

(a) A mapping $L : V \to W$ is linear if the following two properties hold:

(i) $L(v_1 + v_2) = L(v_1) + L(v_2)$ for any vectors $v_1, v_2 \in V$.

(ii) $L(\alpha v) = \alpha L(v)$ for any scalar $\alpha \in \mathbb{R}$ and vector $v \in V$.

(b) If $L$ is linear, we can write that for any scalars $a, b \in \mathbb{R}$ and vectors $v_1, v_2 \in V$, we have

$$L(a v_1 + b v_2) = L(a v_1) + L(b v_2) = a L(v_1) + b (L v_2)$$

where the first and second equalities follow from linearity properties (i) and (ii), respectively.

(c) Using the fact that

$$L(a v_1 + b v_2) = a L(v_1) + b (L v_2) \quad (*)$$

for all scalars $a, b \in \mathbb{R}$ and vectors $v_1, v_2 \in V$, let us separately prove that the linearity properties hold for $L$:

(i) Setting $a = b = 1$ in equation $(*)$ and noting that $1 \cdot v = v \quad \forall v \in V$ and that $1 \cdot w = w \quad \forall w \in W$, we directly obtain that

$$L(1 \cdot v_1 + 1 \cdot v_2) = 1 \cdot L(v_1) + 1 \cdot L(v_2)$$

or equivalently,

$$L(v_1 + v_2) = L(v_1) + L(v_2).$$

(ii) For any scalar $\alpha$ and $v \in V$, setting $a = \alpha$, $b = 0$, $v_1 = v$ in equation $(*)$ and using the facts that $0 \cdot v' = 0_v \quad \forall v' \in V$ and that $0 \cdot w = 0_w \quad \forall w \in W$, we directly obtain that

$$L(\alpha \cdot v + 0 \cdot v_2) = \alpha \cdot L(v) + 0 \cdot L(v_2)$$

or equivalently

$$L(\alpha \cdot v + 0) = \alpha \cdot L(v) + 0_W$$

which is equivalent to

$$L(\alpha v) = \alpha L(v).$$

Thus $L$ is linear.
3. Consider the vector space \((W_3, +, \cdot)\) with

\[ W_3 = \{(x, y, z) : x, y, z \in \mathbb{R} \text{ and } x > 0, y > 0, z > 0\} \]

under the following addition and scalar multiplication operations:

- **Addition:** For any \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) in \(W_3\),

\[ (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1x_2, y_1y_2, z_1z_2). \]

- **Scalar multiplication:** For any scalar \(\alpha \in \mathbb{R}\) and \((x, y, z)\) in \(W_3\),

\[ \alpha \cdot (x, y, z) = (x^\alpha, y^\alpha, z^\alpha). \]

Let \(X_3\) be a subset of \(W_3\) given by

\[ X_3 = \{(x, y, z) \in W_3 : xy = 1\}. \]

(a) Show that \(X_3\) is a vector subspace of \((W_3, +, \cdot)\). (3 pts)

(b) Find a basis for \(X_3\). (6 pts)

(c) Consider the following mapping \(L : X_3 \rightarrow \mathbb{R}^2\) given by

\[ L((x, y, z)) = (\ln(x), \ln(yz)) \quad \forall (x, y, z) \in X_3, \]

where \(\ln(\cdot)\) denotes the natural logarithm and \((\mathbb{R}^2, +, \cdot)\) is the vector space of real-valued ordered pairs under the traditional (component-wise) addition and scalar multiplication operations.

Determine whether or not the mapping \(L\) is linear. (6 pts)

(a) For \(X_3\) to be a vector subspace of \((W_3, +, \cdot)\), the following three conditions have to be met:

(a) The zero vector \(0_{W_3}\) of \(W_3\), i.e. the triple \((1, 1, 1)\) should be an element of \(X_3\),

(b) For any elements \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) of \(X_3\), their sum \((x_1, y_1, z_1) + (x_2, y_2, z_2)\) should also be an element of \(X_3\),

(c) For any real number \(\alpha\) and any element \((x, y, z)\) of \(X_3\), their product \(\alpha \cdot (x, y, z)\) should also be an element of \(X_3\).

Let us now verify these conditions one by one:

(a) For the zero vector \((1, 1, 1)\), indeed we have the product of its first two components is equal to 1 (as each of them is equal to 1); thus the vector \((1, 1, 1)\) satisfies the property of \(X_3\). Hence the zero vector of \(W_3\) belongs to \(X_3\).
(b) Let \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) be elements of \(X_3\), then by definition of \(X_3\), we must have that \(x_1y_1 = 1\) and that \(x_2y_2 = 1\). So \(y_1 = \frac{1}{x_1}\) and \(y_2 = \frac{1}{x_2}\) and we can write

\[
(x_1, y_1, z_1) + (x_2, y_2, z_2) = \left( x_1, \frac{1}{x_1}, z_1 \right) + \left( x_2, \frac{1}{x_2}, z_2 \right) = \left( x_1x_2, \frac{1}{x_1x_2}, z_1z_2 \right)
\]

First note that all the components of the above resulting triple are positive since each of \(x_1, y_1, z_1, x_2, y_2\) and \(z_2\) is positive. Also, let us verify that the product of the first two components of the above vector is equal to 1:

\[
(x_1x_2) \left( \frac{1}{x_1x_2} \right) = \frac{x_1x_2}{x_1x_2} = 1
\]

and thus we indeed have

\[
(x_1, y_1, z_1) + (x_2, y_2, z_2) \in X_3.
\]

(c) Let \(\alpha\) be a real number and let \((x, y, z) = \left( x, \frac{1}{x^\alpha}, z \right)\) be an element of \(X_3\). Then

\[
\alpha \cdot (x, y, z) = \alpha \cdot \left( x, \frac{1}{x^\alpha}, z \right) = \left( x^\alpha, \left( \frac{1}{x} \right)^\alpha, z^\alpha \right) = \left( x^\alpha, \frac{1}{x^\alpha}, z^\alpha \right)
\]

Again all the components of the above resulting triple are positive since each of \(x, y\) and \(z\) is positive. Also, verifying that the product of the first two components of the above vector is equal to 1, we have:

\[
(x^\alpha) \left( \frac{1}{x^\alpha} \right) = \frac{x^\alpha}{x^\alpha} = 1
\]

and therefore we indeed have that

\[
\alpha \cdot (x, y, z) \in X_3.
\]

We conclude that \(X_3\) is a vector subspace of \((W_3, +, \cdot)\).

(b) Any vector \((x, y, z) \in X_3\) satisfies (using the operations of \(W_3\)) the following

\[
(x, y, z) = \left( x, \frac{1}{x}, z \right) = (x, x^{-1}, 1) + (1, 1, z) = (e^{\ln(x)}, e^{-\ln(x)}, 1^{\ln(x)}) + (1^{\ln(z)}, 1^{\ln(z)}, e^{\ln(z)}) = (\ln(x)) \cdot (e, e^{-1}, 1) + (\ln(z)) \cdot (1, 1, e)
\]

where \(e\) is Euler’s number. Setting \(v_1 = (e, e^{-1}, 1)\) and \(v_2 = (1, 1, e)\) and noting that both \(v_1\) and \(v_2\) belong to \(X_3\) (since for each vector, its three components are positive and the product of its first two components equals 1), we have indeed shown above that

\[
(x, y, z) = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2
\]
where $\alpha_1 = \ln(x)$ and $\alpha_2 = \ln(z)$. Thus the set $\{v_1, v_2\}$ is a generating set for $X_3$. We next show that the set $\{v_1, v_2\}$ is linearly independent. Assume that

$$\beta_1 \cdot v_1 + \beta_2 \cdot v_2 = 0$$

and let us show that $\beta_1 = \beta_2 = 0$.

The above equation yields

$$\beta_1 \cdot (e, e^{-1}, 1) + \beta_2 \cdot (1, 1, e) = (1, 1, 1).$$

Thus

$$(e^{\beta_1}, e^{-\beta_1}, 1) + (1, 1, e^{\beta_2}) = (1, 1, 1)$$

and hence

$$(e^{\beta_1}, e^{-\beta_1}, e^{\beta_2}) = (1, 1, 1).$$

Thus

$$\left\{ \begin{array}{l}
  e^{\beta_1} = 1 \\
  e^{-\beta_1} = 1 \\
  e^{\beta_2} = 1
\end{array} \right.$$}

Taking the natural logarithms in each of the above equations yields

$$\left\{ \begin{array}{l}
  \beta_1 = \ln(1) = 0 \\
  -\beta_1 = \ln(1) = 0 \\
  \beta_2 = \ln(1) = 0
\end{array} \right.$$}

Thus $\beta_1 = \beta_2 = 0$ and the set $\{v_1, v_2\}$ is linearly independent.

Therefore $(v_1, v_2, \cdot)$, where $v_1 = (e, e^{-1}, 1)$ and $v_2 = (1, 1, e)$, is a basis of $X_3$.

(c) Recall that in order to show that the mapping $L : X_3 \to \mathbb{R}^2$ is linear we have to show the following two properties:

(i) For any $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ in $X_3$,

$$L((x_1, y_1, z_1) + (x_2, y_2, z_2)) = L((x_1, y_1, z_1)) + L((x_2, y_2, z_2)).$$

(ii) For any $(x, y, z) \in X_3$ and any scalar $\alpha \in \mathbb{R}$,

$$L(\alpha(x, y, z)) = \alpha L((x, y, z)).$$

Let us separately examine each of the above linearity properties:
(i) For any \((x_1, y_1, z_1) = \left(\frac{1}{x_1}, z_1\right)\) and \((x_2, y_2, z_2) = \left(\frac{1}{x_2}, z_2\right)\) in \(X_3\), by properly using the operations of \(W_3\) and \(R^2\), we have the following

\[
L((x_1, y_1, z_1) + (x_2, y_2, z_2)) = L\left(\left(\frac{1}{x_1}, \frac{1}{x_2}, z_1z_2\right)\right) = \left(\ln(x_1x_2), \ln\left(\frac{z_1z_2}{x_1x_2}\right)\right)
\]

On the other hand,

\[
L((x_1, y_1, z_1)) + L((x_2, y_2, z_2)) = L\left(\left(\frac{1}{x_1}, \frac{1}{x_2}, z_1\right)\right) + L\left(\left(\frac{1}{x_1}, \frac{1}{x_2}, z_2\right)\right) = \left(\ln(x_1), \ln\left(\frac{z_1}{x_1}\right)\right) + \left(\ln(x_2), \ln\left(\frac{z_2}{x_2}\right)\right) = \left(\ln(x_1x_2), \ln\left(\frac{z_1z_2}{x_1x_2}\right)\right)
\]

Comparing the above expressions, we directly get that

\[
L((x_1, y_1, z_1) + (x_2, y_2, z_2)) = L((x_1, y_1, z_1)) + L((x_2, y_2, z_2))
\]

and hence this property holds.

(ii) For any \((x, y, z) = \left(\frac{1}{x}, z\right)\) in \(X_3\) and scalar \(\alpha \in \mathbb{R}\), again by properly using the operations of \(W_3\) and \(R^2\), we have the following

\[
L(\alpha(x, y, z)) = L\left(\alpha\left(\frac{1}{x}, \frac{1}{x}, z\right)\right) = L\left(\left(\frac{1}{x}, \frac{1}{x}, z\right)\right) = \left(\alpha \ln(x), \alpha \ln\left(\frac{z}{x}\right)\right)
\]

On the other hand,

\[
\alpha L((x, y, z)) = \alpha L\left(\left(\frac{1}{x}, \frac{1}{x}, z\right)\right) = \alpha \left(\ln(x), \ln\left(\frac{z}{x}\right)\right) = \left(\alpha \ln(x), \alpha \ln\left(\frac{z}{x}\right)\right)
\]
Comparing the above expressions, we directly get that

\[ L(\alpha(x, y, z)) = \alpha L((x, y, z)) \]

and hence this property holds.

We conclude that the mapping \( L \) is linear.
4. Let \((V, +, \cdot)\) and \((W, +, \cdot)\) be real vector spaces, and let \(L : V \to W\) be a linear mapping.

(a) Define what it means for the mapping \(L\) to be injective (i.e., one-to-one). (4 pts)

(b) Give the definition of the kernel of \(L\), \(\text{ker}(L)\). (4 pts)

(c) Show that if \(L\) is injective, then \(\text{ker}(L) = \{0_V\}\). (6 pts)

(d) Show that if \(\text{ker}(L) = \{0_V\}\), then \(L\) is injective. (6 pts)

(a) The mapping \(L : V \to W\) is said to be **injective** (or **one-to-one**) if \(\forall v_1, v_2 \in V\), we have that

\[ v_1 \neq v_2 \implies L(v_1) \neq L(v_2). \]

Equivalently, \(L : V \to W\) is injective if \(\forall v_1, v_2 \in V\), we have that

\[ L(v_1) = L(v_2) \implies v_1 = v_2. \]

(b) The kernel of the linear mapping \(L : V \to W\), \(\text{ker}(L)\), is the set of all \(v\) in \(V\) which are mapped under \(L\) to the zero vector \(0_W\) of \(W\):

\[ \text{ker}(L) = \{v \in V : L(v) = 0_W\}. \]

(c) Given that \(L\) is injective, let us show that if \(v \in \text{ker}(L)\), then \(v = 0_V\) (i.e. the only element of \(\text{ker}(L)\) is the zero vector of \(V\)). Let then \(v \in \text{ker}(L)\). Then, \(L(v) = 0_W\) (by definition of \(v\) being an element of the kernel \(\text{ker}(L)\) of \(L\)). On the other hand, we also know that \(L(0_V) = 0_W\). We therefore have:

\[ L(v) = L(0_V). \]

Since \(L\) is assumed injective, it must follow that \(v = 0_V\). Hence, we have shown that if \(v\) is any element in \(\text{ker}(L)\), then it must follow that \(v = 0_V\); this shows that \(\text{ker}(L) = \{0_V\}\).

(d) Given that \(\text{ker}(L) = \{0_V\}\), let us show that \(L\) is injective. In other words, we have to show that for any \(v_1, v_2 \in V\), the equality \(L(v_1) = L(v_2)\) implies that \(v_1 = v_2\). Let then \(v_1, v_2 \in V\) and assume that \(L(v_1) = L(v_2)\). Hence, by linearity of \(L\):

\[
L(v_1 - v_2) = L(v_1 + (-1)v_2) \\
= L(v_1) + L((-1)v_2) \\
= L(v_1) + (-1)L(v_2) \\
= L(v_1) - L(v_2) \\
= 0_W,
\]

which shows that \(v_1 - v_2 \in \text{ker}(L)\) (since \(L(v_1 - v_2) = 0_W\)). Since we have assumed that \(\text{ker}(L) = \{0_V\}\), it must follow that \(v_1 - v_2 = 0_V\), i.e. that \(v_1 = v_2\). This is what we wanted to show, and this establishes injectivity of \(L\).
5. Let \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \), \( \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \), and \( \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \), and let
\[
A = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix}.
\]

(a) Say what it means for a vector \( \mathbf{v} \) to be an eigenvector of \( A \). (That is, give the definition of “\( \mathbf{v} \) is an eigenvector of \( A \)”.) (3 pts)

(b) Compute \( A\mathbf{v}_1 \), \( A\mathbf{v}_2 \), and \( A\mathbf{v}_3 \). (3 pts)

(c) Your computations in (b) should show that each of \( \mathbf{v}_1 \), \( \mathbf{v}_2 \), and \( \mathbf{v}_3 \) are eigenvectors of \( A \). What are their eigenvalues? (3 pts)

(d) What is \( A^4\mathbf{v}_3 \)? (i.e., the result of putting \( \mathbf{v}_3 \) through \( A \) four times.) (4 pts)

(e) Write the vector \( \mathbf{w} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} \) as a linear combination of \( \mathbf{v}_1 \), \( \mathbf{v}_2 \), and \( \mathbf{v}_3 \). (4 pts)

(f) For a given integer \( n \geq 1 \), give a formula for \( A^n\mathbf{w} \) in terms of the eigenvalues of \( A \). (3 pts)

(a) We say that \( \mathbf{v} \) is an eigenvector of \( A \) if \( \mathbf{v} \) is not equal to the zero vector and there exists a real number \( \lambda \) such that
\[
A\mathbf{v} = \lambda \mathbf{v}.
\]
In this case \( \lambda \) is called the eigenvalue of \( A \) associated with \( \mathbf{v} \).

(b) We have
\[
A\mathbf{v}_1 = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix},
\]

\[
A\mathbf{v}_2 = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}
\]

and
\[
A\mathbf{v}_3 = \begin{pmatrix} 5 & -6 & 3 \\ 2 & -5 & 3 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}.
\]

(c) Indeed,
\[
A\mathbf{v}_1 = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = (-1)\mathbf{v}_1
and hence the eigenvalue of $v_1$ is $\lambda_1 = -1$. Also,

$$A v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = (1) \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = (1)v_2$$

and hence the eigenvalue of $v_2$ is $\lambda_2 = 1$. Finally,

$$A v_3 = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix} = (2) \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = (2)v_3$$

and hence the eigenvalue of $v_3$ is $\lambda_3 = 2$.

(d) Using the fact that $A v_3 = \lambda_3 v_3$ repeatedly, we have

$$A^4 v_3 = A^3(Av_3) = A^3(\lambda_3 v_3) = \lambda_3 A^2(Av_3)$$

$$= \lambda_3 A^2(\lambda_3 v_3) = \lambda_3^2 A(Av_3) = \lambda_3^2 A(\lambda_3 v_3)$$

$$= \lambda_3^3 (Av_3) = \lambda_3^3 (\lambda_3 v_3)$$

$$= \lambda_3^4 v_3$$

Thus

$$A^4 v_3 = \lambda_3^4 v_3 = (2)^4 \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 16 \\ -16 \\ -48 \end{pmatrix}.$$  

(e) To write the vector $w = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$ as a linear combination of $v_1$, $v_2$, and $v_3$, we need to find scalars $\alpha$, $\beta$, and $\gamma$ such that

$$\alpha v_1 + \beta v_2 + \gamma v_3 = w$$

or equivalently

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}. $$

In other words, we have to solve the following system of linear equations:

\[
\begin{cases}
\alpha + \gamma = 5 \\
2\alpha + \beta - \gamma = 4 \\
2\alpha + 2\beta - 3\gamma = -1
\end{cases}
\]

Solving the above system via the Gaussian elimination method, we obtain a unique solution given by:

$$(\alpha, \beta, \gamma) = (4, -3, 1).$$

Thus

$$w = 4v_1 - 3v_2 + v_3.$$
(f) Given integer $n \geq 1$, we deduce that for any eigenvector $v$ with eigenvalue $\lambda$,

$$A^n v = \lambda^n v.$$ 

In other words, if $\lambda$ is an eigenvalue of $A$, then $\lambda^n$ is an eigenvalue of $A^n$. This can be shown iteratively on $n$ using the same procedure as in (d). Thus, using the above fact and the results in (e) and (c), we have

$$A^n w = A^n (4v_1 - 3v_2 + v_3)$$
$$= 4(A^n v_1) - 3(A^n v_2) + (A^n v_3)$$
$$= 4(\lambda^n v_1) - 3(\lambda^n v_2) + (\lambda^n v_3)$$
$$= 4(-1)^n v_1 - 3(1)^n v_2 + (2)^n v_3$$
$$= 4(-1)^n v_1 - 3v_2 + (2)^n v_3.$$
6. Answer the following questions.

(a) Suppose that $A$ and $B$ are invertible $n \times n$ matrices. Is $AB$ invertible? (Provide an argument if your answer is yes, and a counterexample if your answer is no.) (5 pts)

Let $A$ and $B$ be the matrices

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -1 \\ 1 & -4 \\ -2 & 1 \end{pmatrix}.$$

(b) Are either of $A$ or $B$ invertible matrices? (Be sure to give reasons). (2 pts)

(c) Compute the product $AB$. (4 pts)

(d) Is $AB$ an invertible matrix? (4 pts)

(a) For these two $n \times n$ matrices, if $A$ is invertible with inverse $A^{-1}$ and $B$ is invertible with inverse $B^{-1}$, then by associativity of matrix multiplication and the property of the identity matrix $I_n$ (of size $n \times n$), we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n.$$  

Similarly,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$  

Thus $AB$ is invertible and its inverse is given by

$$(AB)^{-1} = B^{-1}A^{-1}.$$  

(b) Since invertibility is defined only for square matrices, noting that $A$ and $B$ are both rectangular matrices (with sizes $2 \times 3$ and $3 \times 2$, respectively), we directly conclude that both $A$ and $B$ are not invertible.

(c) The product $AB$ is given by:

$$\begin{pmatrix} 3 & -2 & 1 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & -4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ -5 & -9 \end{pmatrix}.$$  

(d) $AB$ is invertible iff its determinant is non-zero. We have

$$\det(AB) = ((5)(-9) - (6)(-5)) = -15 \neq 0.$$  

Thus $AB$ is invertible.