INSTRUCTIONS

• This test is 90 MINUTES in length and consists of 4 questions.

• Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

• SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

• No textbook, lecture note, calculator, computer, or other aid, is allowed.

• Good luck!

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1. Consider the system of linear equations given by:

\[
\begin{align*}
    x_1 + 2x_2 - x_3 &= 4, \\
    2x_1 + x_2 &= 2, \\
    -3x_1 - x_2 + x_3 &= 2,
\end{align*}
\]

where we wish to solve for the triple \((x_1, x_2, x_3)\) of real numbers.

(a) **Write the augmented matrix for this system.** (5 pts)

(b) **Transform the augmented matrix to row-echelon form using a sequence of elementary row operations** (clearly indicate which operation you perform at each step). (5 pts)

(c) **Find the set of all solutions of the system of linear equations by applying back-substitution to the system resulting from step (b).** (5 pts)

(a) The given system of linear equations consists of 3 equations and 3 unknowns (namely \(x_1, x_2, x_3\)). The augmented matrix for this system will therefore have 3 rows and \(3+1=4\) columns. Ordering the unknowns by increasing index (i.e. in the order \(x_1, x_2, x_3\)), the augmented matrix \(T\) is therefore given by:

\[
T = \begin{pmatrix}
    1 & 2 & -1 & 4 \\
    2 & 1 & 0 & 2 \\
    -3 & -1 & 1 & 2
\end{pmatrix}.
\]

(b) The augmented matrix \(T\) is not in row-echelon form; we now reduce it to row-echelon form using a sequence of elementary row operations.

(i) The elementary row operation \(-2R_1 + R_2 \rightarrow R_2\) (i.e. adding -2 times row 1 to row 2) yields the augmented matrix:

\[
T = \begin{pmatrix}
    1 & 2 & -1 & 4 \\
    0 & -3 & 2 & -6 \\
    -3 & -1 & 1 & 2
\end{pmatrix}.
\]

(ii) The elementary row operation \(3R_1 + R_3 \rightarrow R_3\) (i.e. adding 3 times row 1 to row 3) yields the augmented matrix:

\[
T = \begin{pmatrix}
    1 & 2 & -1 & 4 \\
    0 & -3 & 2 & -6 \\
    0 & 5 & -2 & 14
\end{pmatrix}.
\]
(iii) The elementary row operation \( \frac{5}{3} R_2 + R_3 \rightarrow R_3 \) (i.e. adding \( \frac{5}{3} \) times row 2 to row 3) yields the augmented matrix:

\[
T = \begin{pmatrix}
1 & 2 & -1 & | & 4 \\
0 & -3 & 2 & | & -6 \\
0 & 0 & \frac{4}{3} & | & 4
\end{pmatrix}.
\]

\( T \) is now in row-echelon form.

(c) The system of linear equations corresponding to the augmented matrix in row-echelon form obtained from the last step of (b) is given by:

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= 4, \\
-3x_2 + 2x_3 &= -6, \\
\frac{4}{3}x_3 &= 4,
\end{align*}
\]

which we solve using back-substitution. Starting from the last equation and solving there for \( x_3 \), we obtain that

\[
x_3 = 3.
\]

Substituting this value of \( x_3 \) in the remaining equations, we obtain (after grouping constants) the system of linear equations:

\[
\begin{align*}
x_1 + 2x_2 &= 7, \\
-3x_2 &= -12,
\end{align*}
\]

and starting again from the last equation, we obtain

\[
x_2 = 4.
\]

We are then left with only one equation, i.e.

\[
x_1 + 2(4) = 7,
\]

from which we obtain

\[
x_1 = -1.
\]

We can therefore conclude that our original system of linear equations has \textbf{a unique solution} given by

\[
(x_1, x_2, x_3) = (-1, 4, 3).
\]
2. (10 pts) Consider the real vector space \((\mathbb{R}^3, +, \cdot)\), and consider the following vectors in \(\mathbb{R}^3\):

\[
\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (2, 2, 2), \quad \mathbf{v}_3 = (-1, 0, 0).
\]

Determine whether or not \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is a generating set for \(\mathbb{R}^3\).

We have to determine whether or not every element of \(\mathbb{R}^3\) is some linear combination of \(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\); if this is the case, then we will have that \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is a generating set for \(\mathbb{R}^3\). Let then \((x, y, z) \in \mathbb{R}^3\) be any element of \(\mathbb{R}^3\), and assume there exist \(\alpha, \beta, \gamma \in \mathbb{R}\) such that \((x, y, z) = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2 + \gamma \cdot \mathbf{v}_3\). Let us solve for \(\alpha, \beta, \gamma\) (if we are able to express \(\alpha, \beta, \gamma\) in terms of \(x, y, z\), then \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is a generating set for \(\mathbb{R}^3\); otherwise, it is not):

\[
(x, y, z) = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2 + \gamma \cdot \mathbf{v}_3 \iff (x, y, z) = \alpha \cdot (1, 0, 1) + \beta \cdot (2, 2, 2) + \gamma \cdot (-1, 0, 0) \\
\iff (x, y, z) = (\alpha + 2\beta - \gamma, 2\beta, \alpha + 2\beta) \\
\iff \begin{cases} 
  x = \alpha + 2\beta - \gamma \\
  y = 2\beta \\
  z = \alpha + 2\beta 
\end{cases}
\]

From the second equation, we obtain that \(\beta = \frac{y}{2}\). Substituting this value of \(\beta\) in the remaining equations yields:

\[
\iff \begin{cases} 
  x = \alpha + y - \gamma \\
  z = \alpha + y 
\end{cases}
\]

From the last equation, we obtain that \(\alpha = z - y\). Substituting this value of \(\alpha\) in the first equation yields \(x = z - y + y - \gamma\), i.e., \(\gamma = z - x\).

Therefore, any element \((x, y, z) \in \mathbb{R}^3\) can be written as \(\alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2 + \gamma \cdot \mathbf{v}_3\), where \(\alpha = z - y\), \(\beta = \frac{y}{2}\) and \(\gamma = z - x\). Hence, \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is a generating set for \(\mathbb{R}^3\).
3. Consider again the real vector space \((\mathbb{R}^3, +, \cdot)\), and let \(W\) be the subset of \(\mathbb{R}^3\) consisting of all triples \((x, y, z)\) of real numbers for which \(2x - z = 0\), i.e.

\[
W = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - z = 0\}.
\]

\(W\) is a vector subspace of \(\mathbb{R}^3\) and hence is itself a real vector space.

(a) **Find a basis for \(W\).** (10pts)

(b) **Using (a), determine the dimension of \(W\).** (2pts)

(a) We have: \(\forall (x, y, z) \in \mathbb{R}^3:\)

\[
(x, y, z) \in W \iff 2x - z = 0 \iff z = 2x \iff (x, y, z) = (x, y, 2x) \iff (x, y, z) = (x, 0, 2x) + (0, y, 0) \iff (x, y, z) = x \cdot (1, 0, 2) + y \cdot (0, 1, 0).
\]

The vectors \(v_1 = (1, 0, 2)\) and \(v_2 = (0, 1, 0)\) are clearly in \(W\); furthermore, the above calculations show that every element of \(W\) is a linear combination of \(v_1, v_2\). Hence \(\{v_1, v_2\}\) is a generating set for \(W\).

Let now \(\alpha, \beta \in \mathbb{R}\) such that \(\alpha \cdot v_1 + \beta \cdot v_2 = (0, 0, 0)\). It then follows that \((\alpha, \beta, 2\alpha) = (0, 0, 0)\), which implies \(\alpha = \beta = 0\). Hence \(\{v_1, v_2\}\) is a linearly independent subset of \(W\).

We have shown that \(\{v_1, v_2\}\) is a generating set for \(W\) and that it is a linearly independent subset of \(W\); it follows therefore that \((v_1, v_2)\) is a basis for \(W\).

(b) The basis \((v_1, v_2)\) of \(W\) found in (a) has 2 elements; it follows therefore that \(W\) is a vector space of dimension 2.
4. (8 pts) Let \((V, +, \cdot)\) be a real vector space. Let \(v_1, v_2\) be two distinct elements of \(V\), and assume \(\{v_1, v_2\}\) is a generating set for \(V\). Let now \(v_3 = v_1 + v_2\) and \(v_4 = v_1 - v_2\). Show that \(\{v_3, v_4\}\) is also a generating set for \(V\).

Note first that \(v_3 + v_4 = 2v_1\), which is equivalent to
\[
v_1 = \frac{1}{2} \cdot v_3 + \frac{1}{2} \cdot v_4.
\]
Also, we have that \(v_3 - v_4 = 2v_2\), which is equivalent to
\[
v_2 = \frac{1}{2} \cdot v_3 - \frac{1}{2} \cdot v_4.
\]
By assumption that \(\{v_1, v_2\}\) is a generating set for \(V\), every element of \(V\) can be written as a linear combination of \(v_1\) and \(v_2\). Let then \(v \in V\) be any element of \(V\); there exists \(\alpha, \beta \in \mathbb{R}\) such that
\[
v = \alpha \cdot v_1 + \beta \cdot v_2,
\]
which, using the expressions for \(v_1\) and \(v_2\) above, yields:
\[
v = \alpha \cdot \left( \frac{1}{2} \cdot v_3 + \frac{1}{2} \cdot v_4 \right) + \beta \cdot \left( \frac{1}{2} \cdot v_3 - \frac{1}{2} \cdot v_4 \right)
\]
\[
= \left( \frac{\alpha}{2} + \beta \right) \cdot v_3 + \left( \frac{\alpha}{2} - \frac{\beta}{2} \right) \cdot v_4.
\]
This shows that \(v\) is also a linear combination of \(v_3\) and \(v_4\).
We have shown therefore that every element of \(V\) can be written as a linear combination of \(v_3\) and \(v_4\); this shows that \(\{v_3, v_4\}\) is a generating set for \(V\).