INSTRUCTIONS

- This test is 90 MINUTES in length and consists of 4 questions.

- Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

- SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

- No textbook, lecture note, calculator, computer, or other aid, is allowed.

- Good luck!

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1. Consider the system of linear equations given by:

\[
\begin{align*}
  x_1 - x_2 + 2x_3 &= 3, \\
  -3x_1 - 3x_2 &= 6, \\
  -2x_1 - 2x_3 &= -1,
\end{align*}
\]

where we wish to solve for the triple \((x_1, x_2, x_3)\) of real numbers.

(a) Write the augmented matrix for this system. (2 pts)

(b) Transform the augmented matrix to row-echelon form using a sequence of elementary row operations (clearly indicate which operation you perform at each step). (5 pts)

(c) Find the set of all solutions of the system of linear equations by applying back-substitution to the system resulting from step (b). (3 pts)

(a) Solution: The augmented matrix is given by

\[
\begin{pmatrix}
  1 & -1 & 2 & 3 \\
  -3 & -3 & 0 & 6 \\
  -2 & 0 & -2 & -1
\end{pmatrix}
\]

(b) Solution: We perform row operations as follows.

\[
\begin{pmatrix}
  1 & -1 & 2 & 3 \\
  -3 & -3 & 0 & 6 \\
  -2 & 0 & -2 & -1
\end{pmatrix} \xrightarrow{R_2 + 3R_1 \rightarrow R_2} \begin{pmatrix}
  1 & -1 & 2 & 3 \\
  0 & -6 & 6 & 15 \\
  -2 & 0 & -2 & -1
\end{pmatrix} \xrightarrow{R_3 + 2R_1 \rightarrow R_3} \begin{pmatrix}
  1 & -1 & 2 & 3 \\
  0 & -6 & 6 & 15 \\
  0 & 2 & 2 & 5
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & -1 & 2 & 3 \\
  0 & -6 & 6 & 15 \\
  0 & 2 & 2 & 5
\end{pmatrix} \xrightarrow{\frac{1}{3}R_2 \rightarrow R_2} \begin{pmatrix}
  1 & -1 & 2 & 3 \\
  0 & -2 & 2 & 5 \\
  0 & 2 & 2 & 5
\end{pmatrix} \xrightarrow{R_3 - R_2 \rightarrow R_3} \begin{pmatrix}
  1 & -1 & 2 & 3 \\
  0 & -2 & 2 & 5 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]
1. (continued)

(c) Solution: This augmented matrix is equivalent to the linear equations

\[
\begin{align*}
  x_1 - x_2 + 2x_3 &= 3 \\
  -2x_2 + 2x_3 &= 5 \\
\end{align*}
\]

Solving for \(x_2\) in the second equation yields

\[
x_2 = x_3 - \frac{5}{2}
\]

and substituting this into the first equation gives

\[
x_1 = x_2 - 2x_3 + 3 = x_3 - \frac{5}{2} - 2x_3 + 3 = -x_3 + \frac{1}{2}
\]

and so our solutions are

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} = \begin{pmatrix}
  -x_3 + \frac{1}{2} \\
  x_3 - \frac{5}{2} \\
  x_3
\end{pmatrix} = x_3 \cdot \begin{pmatrix}
  -1 \\
  1 \\
  0
\end{pmatrix} + \begin{pmatrix}
  \frac{1}{2} \\
  -\frac{5}{2} \\
  0
\end{pmatrix}
\]
2. (8 pts) Consider the real vector space $\mathbb{R}^3$, and consider the following vectors in $\mathbb{R}^3$:

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\
v_2 &= \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \\
v_3 &= \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} \\
v_4 &= \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix}
\end{align*}
\]

Determine whether or not \{v_1, v_2, v_3, v_4\} is a generating set for $\mathbb{R}^3$.

**Solution:** We want to check whether or not it is possible to write, for arbitrary \(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3\),

\[
\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix}
\]

for some appropriate choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. This is equivalent to solving the equivalent system of linear equations, which we can do with the augmented matrix

\[
\begin{pmatrix}
1 & 3 & 5 & 0 & \vdots & a \\
-1 & -2 & -4 & -1 & \vdots & b \\
2 & -2 & 2 & 8 & \vdots & c
\end{pmatrix}
\]

So let us row reduce this matrix. We find

\[
\begin{pmatrix}
1 & 3 & 5 & 0 & \vdots & a \\
0 & 1 & 1 & -1 & \vdots & b + a \\
0 & -8 & -8 & 8 & \vdots & c - 2a
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 3 & 5 & 0 & \vdots & a \\
0 & 1 & 1 & -1 & \vdots & b + a \\
0 & 0 & 0 & 0 & \vdots & c + 6a + 8b
\end{pmatrix}
\]

Since the last row is equivalent to the equality

\[
0 = c + 6a + 8b
\]

we find that this does not have any solutions unless this equation holds among the coefficients $a, b, c$. In particular, for the vector \(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\) this does not hold, and so this vector is not in the span of $v_1, v_2, v_3, v_4$ i.e. they are not a generating set for $\mathbb{R}^3$. 
3. Consider the vector space $\mathbb{R}^4$ consisting of 4-tuples $(x, y, z, w)$ of real numbers, and consider the subspace

$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x - y = 0, 2z + w = 0\}$

(Note: You do not need to verify that this is a subspace).

(a) **Find a basis for $W$, and prove that it is a basis.** (8pts)

(b) **Using (a), determine the dimension of $W$.** (2pts)

(a) **Solution:** Let the vector

$$v = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in W$$

From the definition of $W$, we see that the coefficients must satisfy

$$x = y \quad \text{and} \quad w = -2z$$

and so we can rewrite $v$ as

$$v = \begin{pmatrix} x \\ x \\ z \\ -2z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \\ -2z \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$

I claim that these two vectors (which we will denote by $w_1, w_2$) are a basis. From the computation above (as $v$ was arbitrary) it follows that they span $W$, so it only remains to check that they are linearly independent.

Suppose then that $\alpha w_1 + \beta w_2 = 0$. This is equivalent to

$$\begin{pmatrix} \alpha \\ \alpha \\ \beta \\ -2\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Comparing the first and third (say) coefficients immediately yields that $\alpha = 0 = \beta$, and so these are linearly independent, hence a basis.

(b) **Solution:** Since the basis consists of two vectors, the dimension of $W$ is 2.
4. (6pts) Let $A$ be a $2 \times 3$ matrix. Recall that the \textit{Null Space} of $A$ is the set of all vectors $x \in \mathbb{R}^3$ such that

$$Ax = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

\textbf{Prove that the Null Space of $A$ is a subspace of $\mathbb{R}^3$.}

(Hint: Remember the three conditions that you must check to show that something is a subspace!)

\textbf{Solution:} We must verify three things. First, we will verify that $0 \in \ker A$. Suppose that

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

then an easy calculation shows that

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as claimed, and so $0 \in \ker A$.

Suppose now that $x, y \in \ker A$. That is, $Ax = 0 = Ay$. Then it follows that

$$A(x + y) = Ax + Ay = 0 + 0 = 0$$

where the first equality is due to the linearity of multiplying by the matrix $A$. Hence it follows that $x + y \in \ker A$ as claimed.

Lastly, suppose that $\alpha \in \mathbb{R}, x \in \ker A$. We want to show that $\alpha x \in \ker A$ or equivalently that $A(\alpha x) = 0$. We compute that

$$A(\alpha x) = \alpha (Ax) = \alpha 0 = 0$$

where again, the first equality is due to the linearity of $A$. It follows that $\ker A$ is a subspace as claimed.