INSTRUCTIONS

• This test is 90 MINUTES in length and consists of 4 questions.

• Answer all questions, writing clearly in the space provided. If you need more room, continue to answer on the back of the previous page, providing clear directions to the marker.

• SHOW ALL YOUR WORK, clearly and in order, if you wish to receive full credit.

• No textbook, lecture notes, calculator, computer, or other aid, is allowed.

• Good luck!

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1. (10 pts) For each of the following matrices, write down a basis for

(i) The image of the matrix, \( \text{im} \ A \)

(ii) The kernel of the matrix, \( \text{ker} \ A \)

Lastly, verify that the rank-nullity theorem holds in each case.

(a) \( A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 4 & 3 \end{pmatrix} \)

Solution: We row reduce this as

\[
\begin{pmatrix} 0 & 1 & -2 \\ -1 & 4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 4 & 3 \\ 0 & 1 & -2 \end{pmatrix}
\]

with the leading coefficients in red. It follows then that the image has the basis

\[
\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}
\]

For the kernel, we let \( x_3 = s \) and note that the second row reads \( x_2 = 2x_3 = 2s \), while the first row reads \( 0 = -x_1 + 4x_2 + 3x_3 = -x_1 + 11s \). Since any element in the kernel can be written

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11s \\ 2s \\ s \end{pmatrix}
\]

Thus our basis for the kernel is

\[
\begin{pmatrix} 11 \\ 2 \\ 1 \end{pmatrix}
\]

Finally, we see that the rank is 2, the nullity is 1. That is,

\[ 1 + 2 = 3 \]
(b) \( A = \begin{pmatrix} 0 & 0 & -1 & 1 & 2 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 4 & 2 & 2 & 4 \end{pmatrix} \)

**Solution:** We row reduce to find

\[
\begin{pmatrix} 0 & 0 & -1 & 1 & 2 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 4 & 2 & 2 & 4 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 4 & 2 & 2 & 4 \end{pmatrix} \xrightarrow{R_2 - 2R_1 \rightarrow R_3} \begin{pmatrix} 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & -2 & 2 & 4 \end{pmatrix} \xrightarrow{R_3 - 2R_2 \rightarrow R_3} \begin{pmatrix} 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

It follows that a basis for the image is

\[
\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}
\]

For the kernel, we note that the first, fourth, and fifth columns have no leading coefficients, so we let \( x_1 = r, x_4 = s, x_5 = t \). Then the given rows read

\[
2x_2 + 2x_3 = 0 \quad -x_3 + x_4 + 2x_5 = 0
\]

and so we find that \( x_3 = s + 2t \) and that \( x_2 = -s - 2t \). Since any element in the kernel can be written as

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} r \\ -s - 2t \\ s + 2t \\ s \\ t \end{pmatrix}
\]

It follows then that a basis for the kernel is given by the vectors

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}
\]

Finally, as the rank is 2 and the nullity is 3, we see that

\[
2 + 3 = 5
\]

which is the number of columns of the matrix.
2. (12 pts) Consider the following matrices:

\[
A = \begin{pmatrix}
2 & 2 & 2 & -2 \\
-3 & 0 & 0 & -1
\end{pmatrix} \quad B = \begin{pmatrix}
1 & -2 \\
4 & -1
\end{pmatrix} \quad C = \begin{pmatrix}
5 & 3 \\
-2 & 0 \\
0 & 0
\end{pmatrix} \quad D = \begin{pmatrix}
1 & 0 \\
6 & 7 \\
-6 & 7 \\
0 & 1
\end{pmatrix}
\]

Compute each of the following products, or state why they do not make sense.

(a) \( AD \)

Solution: Since \( A \) is \( 2 \times 4 \) and \( D \) is \( 4 \times 2 \), the product does make sense. We compute it to be

\[
AD = \begin{pmatrix}
2 & 26 \\
-3 & -1
\end{pmatrix}
\]

(b) \( A^2 \)

Solution: Since \( A^2 = A \cdot A \) and the number of rows of \( A \) does not match the number of columns, this product does not make sense.

(c) \( B^2 \)

Solution: This product does make sense, and we compute it to be

\[
B^2 = \begin{pmatrix}
-7 & 0 \\
0 & -7
\end{pmatrix}
\]
(d) $DA$

Solution: This product also makes sense, since the number of columns of $D$ is equal to the number of rows of $A$. We compute this as

$$DA = \begin{pmatrix}
2 & 2 & 2 & -2 \\
-9 & 12 & 12 & -19 \\
-33 & -12 & -12 & 5 \\
-3 & 0 & 0 & -1
\end{pmatrix}$$

(e) $AC$

Solution: This does not make sense, and $A$ has 4 columns and $C$ has only three rows.

(f) $CA$

Solution: This matrix product makes sense, and it is

$$CA = \begin{pmatrix}
1 & 10 & 10 & -13 \\
-4 & -4 & -4 & 4 \\
0 & 0 & 0 & 0
\end{pmatrix}$$
3. (9pts) By computing the inverse of the appropriate $2 \times 2$ matrix, solve the following system of linear equations for each of the given values of $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

\begin{align*}
4x_1 - 3x_2 &= b_1 \\
7x_1 - 5x_2 &= b_2
\end{align*}

a) $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution: We first write the system of equations as

\[
\begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\]

and we compute the inverse of the $2 \times 2$ matrix to be

\[
\begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}^{-1} = \begin{pmatrix} -5 & 3 \\ -7 & 4 \end{pmatrix}
\]

Thus in this first case, we have that

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}
\]

b) $b = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

Solution: Using the same matrix and inverse as before, we find that

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 21 \end{pmatrix}
\]

c) $b = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Solution: Using the same matrix and inverse as before, we find that

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ -18 \end{pmatrix}
\]
4. (6pts) Let \( A \) be an \( n \times m \) matrix and let \( B \) be an \( m \times r \) matrix. Suppose that

\[
AB = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}
\]

a) (2pts) Show that \( ABx = 0 \) for all \( x \in \mathbb{R}^r \).

Solution: Since all of the entries of the matrix \( AB \) are zero, we compute

\[
AB \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} 0x_1 + 0x_2 + \cdots + 0x_r \\ \vdots \\ 0x_1 + 0x_2 + \cdots + 0x_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\]

i.e. no matter the input \( x \), we find that \( ABx = 0 \) as claimed.

b) (4pts) Show that \( \text{im } B \subset \ker A \).

Solution: We want to show than any \( x \in \text{im } B \) must also be in \( \ker A \). That is, we want to show for any such \( x \), that

\[
Ax = 0
\]

So let \( x \in \text{im } B \). This means that there is some \( y \) so that \( x = By \). Let us now apply \( A \) to both sides of this equality. We get

\[
Ax = ABy = 0
\]

by the part (a) of the problem. But this shows that \( x \) is in the kernel as claimed.