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# STABILIZATION OF SYSTEMS WITH ONE DEGREE OF UNDERACTUATION WITH ENERGY SHAPING: A GEOMETRIC APPROACH*

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**Abstract.** A geometric formulation for stabilization of systems with one degree of underactuation which fully solves the energy shaping problem for these system is given. The results show that any linearly controllable simple mechanical system with one degree of underactuation is stabilizable by energy shaping, even in the absence of gyroscopic force, via a closed-loop metric which is not necessarily positive-definite. An example of a system with one degree of underactuation is provided for which the stabilization by energy shaping method is not achievable, in the absence of gyroscopic forces, using a positive-definite closed-loop metric.

**Key words.** energy shaping, stabilization of mechanical systems, geometric control

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**1. Introduction.** One of the recent developments for stabilization of simple mechanical systems is stabilization by energy shaping method. The central idea concerns the construction of a feedback for which the closed-loop system inherits the structure of a mechanical system. If such a feedback exists, the stability of the equilibria can be guaranteed if the Hessian of the closed-loop potential function is positive-definite. An important feature of the method, in case it is applicable, is providing a procedure that allows the design of nonlinear stabilizing feedbacks.

The first classical appearance of the notion of potential energy shaping problem is in [38]. The investigation on the capabilities of this method continued in both Hamiltonian framework [40, 39, 29, 16] and Lagrangian framework [10, 9, 7, 8] by looking at the properties of interconnected mechanical systems and introducing the concept of kinetic energy shaping. In Hamiltonian framework, the method is modified into the IDA-PBC method by including the notion of kinetic energy shaping; see [31] for a summary. The equivalent version of the IDA-PBC method in Lagrangian framework, the so-called controlled Lagrangian method, is initiated and developed in [25, 8, 7]; the equivalence of these two methods has been shown in [15, 6]. Extension to shaping both energy and external forces has been investigated recently in [12]. In recent work, it has been realized that the space of possible kinetic energy feedbacks can be enlarged by considering the addition of appropriate gyroscopic forcing [41].

It turns out that a necessary condition for stabilization of simple mechanical control systems by energy shaping method is linearly controllability [42, 34]. In both methods, the question of energy shaping for a mechanical system reduces to solving a nonlinear system of partial differential equations. Thus a large number of papers on energy shaping method deal with finding a parametrization of solutions to this system of partial differential equations for a particular class of mechanical systems; see, for

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example, [43, 32, 30]. A differential geometric approach to the kinetic energy shaping problem—the so-called $\lambda$-method—has been presented in [5, 3, 4]. A system of linear partial differential equations is proposed for the kinetic energy shaping problem in terms of a new variable, $\lambda = \mathbb{G}_{\mathrm{cl}}^{\sharp}\mathbb{G}_{\mathrm{ol}}^{\flat}$, where $\mathbb{G}_{\mathrm{ol}}$ and $\mathbb{G}_{\mathrm{cl}}$ are the open-loop and closed-loop metrics, respectively. The main idea of the $\lambda$-method is that it transforms the set of quasi-linear equations for kinetic energy shaping into a set of overdetermined linear partial differential equations [3]. This method is extended to systems with gyroscopic forces; see [13].

In [27] an affine differential geometric approach to energy shaping has been introduced. In recent work, sufficient conditions for the existence of potential energy shaping are derived assuming that kinetic energy shaping has been performed [28]. The results are based on the integrability theory for linear partial differential equations developed in [19, 37]. In [18], the properties of the kinetic energy shaping partial differential equations has been explored using jet bundle theory and Spencer–Goldschmidt integrability techniques. The authors provide a set of sufficient conditions for the kinetic energy shaping. Moreover, these results are coupled with the integrability results of potential energy shaping [28] in order to provide a general approach for the total energy shaping. The results of the current paper fully rely on the integrability analysis of energy shaping partial differential equations and clarify the importance of such results.

Numerous systems considered in the literature on energy shaping have one degree of underactuation. In [2] the authors partially show that, under some conditions, systems with one degree of underactuation can be stabilized using energy shaping feedback. The results rely on a transformation of the system of partial differential equations and do not give any geometric insight into the energy shaping partial differential equations. In [18] the authors show that if $\Sigma_{\mathrm{ol}}$ is a simple mechanical control system with one degree of underactuation, for each bundle automorphism which satisfies the $\lambda$-equation [5, 3], there exists a closed-loop metric and a closed-loop potential function which satisfy the energy shaping system of partial differential equations. This result is independently proved in [14] using the Cauchy–Kowalevski theorem. In the absence of gyroscopic forces, it is not clear if these existence results guarantee that the solutions are stabilizing ones. In fact, as we will demonstrate in this paper, in the absence of gyroscopic forces, the existence of solutions to the kinetic energy shaping does not guarantee the existence of a necessarily positive-definite closed-loop metric which gives rise to a stabilizing energy shaping feedback.

In the current paper, we show that a linearly controllable simple mechanical control system with one degree of underactuation can be stabilized using an energy shaping feedback, possibly via a closed-loop metric which is not positive-definite. This paper is organized as follows. In section 2 we recall the affine geometric formulation of the energy shaping problem. We give a summary of the integrability results for the partial differential equations in potential energy shaping in section 3. We, directly and without proof, use the results of [18] and [28]; thus we do not review the main integrability theorem of Goldschmidt [19]. A reader interested in understanding the formal integrability of partial differential equations is referred to [19, 20, 21, 22, 37, 33, 23, 24, 36]. Furthermore, we recall the $\lambda$-method for kinetic energy shaping problem. Section 4 contains the main contribution of this paper: we show that all linearly controllable simple mechanical control systems with one degree of underactuation are stabilizable using energy shaping method. We fully characterize a set of solutions to the kinetic energy shaping problem which is large enough to guarantee the stabilization by energy shaping method.

**Notation.** The differential geometric notions used in modeling of simple mechanical systems are assumed here, and the unfamiliar reader is referred to [11, 1, 26] for more details. The identity map for a set $S$ is denoted by $\mathrm{id}_S$ and the image of a map $f : S \to W$ by $\mathsf{Im}(f)$. For a vector space $\mathsf{V}$ the set of $(r, s)$-tensors on $\mathsf{V}$ is denoted by $\mathrm{T}^r_s(\mathsf{V})$. By $\mathsf{S}_k\mathsf{V}$ and $\Lambda_k\mathsf{V}$ we denote, respectively, the set of symmetric and skew-symmetric $(0, k)$-tensors on $\mathsf{V}$. Let $\mathsf{A}$ be a $(0, 2)$-tensor on $\mathsf{V}$. We define the flat map $\mathsf{A}^\flat : \mathsf{V} \to \mathsf{V}^*$ by $\langle \mathsf{A}^\flat(u); v\rangle = \mathsf{A}(u, v)$, $u, v \in \mathsf{V}$. The inverse of the flat map is denoted by $\mathsf{A}^\sharp : \mathsf{V}^* \to \mathsf{V}$ in case $\mathsf{A}^\flat$ is invertible. We also define a similar notation for a $(0, 3)$-tensor $\mathsf{A}$ on $\mathsf{V}$ by

$$\langle \mathsf{A}^\flat(u), w\rangle = \mathsf{A}(w, u, u), \qquad u, w \in \mathsf{V}.$$

For $\mathsf{S} \subset \mathsf{V}$ and $\mathsf{W} \subset \mathsf{V}^*$ we denote

$$\begin{aligned} \mathrm{ann}(\mathsf{S}) &= \{\alpha \in \mathsf{V}^* \mid \alpha(v) = 0 \quad \forall\, v \in \mathsf{S}\},\\ \mathrm{coann}(\mathsf{W}) &= \{v \in \mathsf{V} \mid \alpha(v) = 0 \quad \forall\, \alpha \in \mathsf{W}\}. \end{aligned}$$

We denote by $(\mathsf{E}, \pi, \mathsf{Q})$ a fibered manifold $\pi : \mathsf{E} \to \mathsf{Q}$. The *vertical bundle* of the fibered manifold $\pi$ is the subbundle of $\mathsf{T}\pi : \mathsf{TE} \to \mathsf{TQ}$ given by $\mathsf{V}\pi = \ker(\mathsf{T}\pi)$. We denote by $\mathsf{J}_k\pi$ the *bundle of k-jets* [35]. A local section of $\pi$ is a pair $(U, \xi)$, where $U$ is an open submanifold of $\mathsf{Q}$ and $\xi$ is a map $\xi : U \to \mathsf{E}$ such that $\pi \circ \xi = \mathrm{id}_U$. If $(\xi, U)$ is an analytic local section of $\pi$, we denote its $k$-jet by $j_k\xi$. We denote an element of $\mathsf{J}_k\pi$ by $j_k\xi(x)$, where $x \in U$. For more information about geometric properties of jet bundles, see [35]. A *partial differential equation* is a fibered submanifold $\mathsf{R}_k \subset \mathsf{J}_k\pi$. The Goldschmidt theorem investigates the conditions under which one can construct formal solutions of a given partial differential equation by constructing their Taylor series order by order [19].

**2. Statement of the energy shaping problem.** We assume that the reader is familiar with the affine geometric setup for simple mechanical systems [11]. A *forced simple mechanical system* is a quadruple $\Sigma = (\mathsf{Q}, \mathbb{G}, V, \mathcal{F}_{\mathrm{e}})$, where $\mathsf{Q}$ is an $n$-dimensional manifold called the *configuration manifold*, $\mathbb{G}$ is a Riemannian metric on $\mathsf{Q}$, $V$ is a function on the configuration manifold called the *potential function*, and $\mathcal{F}_{\mathrm{e}} : \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q}$ is a bundle map over $\mathrm{id}_{\mathsf{Q}}$ called the *external force*. We denote by $\nabla^{\mathbb{G}}$ the covariant derivative with respect to the associated Levi–Civita connection. The governing equations for a forced simple mechanical system are

$$\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}^\sharp \circ dV(\gamma(t)) + \mathbb{G}^\sharp\mathcal{F}_{\mathrm{e}}(\gamma'(t)),$$

where $\gamma : I \to \mathsf{Q}$ is an analytic curve on $\mathsf{Q}$.

Similarly, a *simple mechanical control system* is a quintuple $\Sigma = (\mathsf{Q}, \mathbb{G}, V, \mathcal{F}_{\mathrm{e}}, \mathcal{W})$, where $\mathsf{Q}$ is an $n$-dimensional manifold called the *configuration manifold*, $\mathbb{G}$ is a Riemannian metric on $\mathsf{Q}$, $V$ is a function on the configuration manifold called the *potential function*, $\mathcal{F}_{\mathrm{e}} : \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q}$ is a bundle map over $\mathrm{id}_{\mathsf{Q}}$ called the *external force*, and $\mathcal{W}$ is a subbundle of $\mathsf{T}^*\mathsf{Q}$ called the control subbundle [11]. The governing equations for a simple mechanical control system are

$$\nabla^{\mathbb{G}}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}^\sharp \circ dV(\gamma(t)) + \mathbb{G}^\sharp\mathcal{F}_{\mathrm{e}}(\gamma'(t)) + \mathbb{G}^\sharp u(\gamma'(t)),$$

where $\gamma : I \to \mathsf{Q}$ is a curve on $\mathsf{Q}$ and $u : \mathsf{TQ} \to \mathcal{W}$ is the control force.

Given an open-loop simple mechanical control system $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathcal{F}_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$, we seek a control force such that the closed-loop system is a forced simple mechanical system $\Sigma_{\text{cl}} = (\mathsf{Q}, \mathbb{G}_{\text{cl}}, V_{\text{cl}}, \mathcal{F}_{\text{cl}})$, possibly with some external force. The reason for seeking this as the closed-loop system is that the stability analysis of the equilibria for mechanical systems is well understood [11, Chapter 6]. In this paper we assume that the open-loop external force $\mathcal{F}_{\text{ol}}$ is zero.

It is well known that the presence of gyroscopic forces enlarges the space of possible closed-loop metrics while it does not change the total energy of the closed-loop system. As we see in the next sections, systems with one degree of underactuation can be stabilized using energy shaping feedback without gyroscopic forces, and thus our statement of energy shaping problem in this paper does not involve gyroscopic forces. For more details on energy shaping at the presence of these forces, see [41, 13, 27]. In the following, we present the statement of the energy shaping problem in the absence of gyroscopic forces.

Definition 2.1 (energy shaping problem in the absence of gyroscopic forces). *Let $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}_{\text{ol}}, V_{\text{ol}}, \mathcal{F}_{\text{ol}}, \mathcal{W}_{\text{ol}})$ be an open-loop simple mechanical control system with $\mathcal{F}_{\text{ol}} = 0$. If there exists a bundle map $u_{\text{shp}} : \mathsf{TQ} \to \mathcal{W}_{\text{ol}}$ (called control) with $u_{\text{shp}} = -u_{\text{kin}} - u_{\text{pot}}$ such that the closed-loop system is a forced simple mechanical system $\Sigma_{\text{cl}} = (\mathsf{Q}, \mathbb{G}_{\text{cl}}, V_{\text{cl}}, 0)$ and*

1. $\mathbb{G}^{\sharp}_{\text{ol}} \circ u_{\text{kin}}(\gamma'(t)) = \nabla^{\mathbb{G}_{\text{cl}}}_{\gamma'(t)}\gamma'(t) - \nabla^{\mathbb{G}_{\text{ol}}}_{\gamma'(t)}\gamma'(t)$,
2. $u_{\text{pot}}(\gamma(t)) = \mathbb{G}^{\flat}_{ol} \circ \mathbb{G}^{\sharp}_{cl} dV_{\text{cl}}(\gamma(t)) - dV_{\text{ol}}(\gamma(t))$,

*then the control $u_{\text{shp}}$ is called an energy shaping feedback.*

Throughout this work, we assume that the equilibrium point $q_0 \in \mathsf{Q}$ is a regular point for $\mathcal{W}_{\text{ol}}$. Moreover, we assume that the control codistribution $\mathcal{W}_{\text{ol}}$ is *integrable.* This assumption is common in the literature and many examples fall into this case. The conditions of Definition 2.1 contain as unknowns the closed-loop metric $\mathbb{G}_{\text{cl}}$. One can observe that these equations involve the first jet of the unknowns. One can construct concretely a set of first-order partial differential equations as necessary and sufficient conditions for the existence of an energy shaping feedback. Let $\mathcal{W} \subset \mathsf{T}^*\mathsf{Q}$ be a given subbundle and define the associated $\mathbb{G}_{\text{ol}}$-orthogonal projection map $P \in \Gamma^{\omega}(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ by

$$\mathsf{Ker}(P) = \mathbb{G}^{\sharp}_{\text{ol}}\mathcal{W}.$$

Note that $P$ completely prescribes $\mathcal{W}$. We apply $P$ to the equation from part 2 of Definition 2.1 to arrive at the following equation:

$$P(\nabla^{\mathbb{G}_{\text{cl}}}_{\gamma'(t)}\gamma'(t) - \nabla^{\mathbb{G}_{\text{ol}}}_{\gamma'(t)}\gamma'(t)) = 0.$$

Assume $\mathsf{Q}$ is an $n$-dimensional manifold and $\mathcal{W}$ is an integrable codistribution of dimension $n - m$. In adapted local coordinates the kinetic energy shaping partial differential equation is given by

$$P^a_r(\mathbb{G}^{rl}_{\text{cl}}(\mathbb{G}_{\text{cl},lj,k} + \mathbb{G}_{\text{cl},lk,j} - \mathbb{G}_{\text{cl},kj,l}) - \mathbb{G}^{rl}_{\text{ol}}(\mathbb{G}_{\text{ol},lj,k} + \mathbb{G}_{\text{ol},lk,j} - \mathbb{G}_{\text{ol},kj,l})) = 0, \tag{2.1}$$

where $i, j, k, l, r \in \{1, \ldots, n\}$, $a \in \{1, \ldots, m\}$ and we denote the first derivative of $\mathbb{G}_{\text{cl}lj}$ with respect to $q^k$ by $\mathbb{G}_{\text{cl},lj,k}$. Similarly, let $\hat{P} : \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}/\mathcal{W}_{\text{ol}}$ be the canonical projection onto the quotient vector bundle. We have

$$\hat{P}(\mathbb{G}^{\flat}_{\text{ol}} \circ \mathbb{G}^{\sharp}_{\text{cl}} dV_{\text{cl}}(\gamma(t)) - dV_{\text{ol}}(\gamma(t))) = 0.$$

In local coordinates we have

$$\hat{P}^i_a(\mathbb{G}_{\text{ol},ij}\mathbb{G}_{\text{cl}}{}^{jk}V_{\text{cl},k} - V_{\text{ol},k}) = 0, \tag{2.2}$$

where $i, j, k \in \{1, \dots, n\}$, $a \in \{1, \dots, m\}$ and we denote the first derivative of $V_{\mathrm{cl}}$ with respect to $q^k$ by $V_{\mathrm{cl},k}$. For more details on the affine differential geometric setup of energy shaping problem, see [27].

## 3. Summary of some integrability results.

In [28] the potential energy shaping partial differential equation has been shown to be formally integrable under a surjectivity condition. An important corollary of this is that the choice of $\mathbb{G}_{\mathrm{cl}}$ affects the set of solutions that one might get for potential energy shaping. A bad choice of $\mathbb{G}_{cl}$ might make it impossible to find any potential energy shaping feedback. In a recent paper [18], the authors show that the system of partial differential equations for kinetic energy shaping is formally integrable under a surjectivity condition. Moreover, they investigate the obstruction for integrability of the total energy shaping partial differential equations. Since the integrability conditions of the potential energy shaping partial differential equations is an integral part of Theorem 4.3, we review this result in this section without mentioning the proofs. Furthermore, we briefly recall the $\lambda$-method for proceeding with the kinetic energy shaping. We refer the interested reader to [18] for more details on the integrability character of the partial differential equations in $\lambda$-method.

### 3.1. Potential energy shaping.

In this section, we explore aspects of potential energy shaping. We recall the results for potential energy shaping after kinetic energy shaping from [28]. Denote the bundle automorphism $\mathbb{G}_{\mathrm{ol}}^{\flat} \circ \mathbb{G}_{\mathrm{cl}}^{\sharp}$ by $\Lambda_{\mathrm{cl}}$. Define a codistribution $\mathcal{W}_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1}(\mathcal{W}_{\mathrm{ol}})$ and assume that this codistribution is integrable. Let $\mathsf{PS} \doteq (\mathsf{Q} \times \mathbb{R}, \pi, \mathsf{Q})$ be the trivial vector bundle over $\mathsf{Q}$, so that a section of $\pi$ corresponds to a potential function via the formula $q \mapsto (q, V(q))$. We define a $\mathsf{T}^*\mathsf{Q}$-valued differential operator $\mathfrak{D}_{\mathsf{d}}(V) = \mathsf{d}V$, which induces a vector bundle map $\Phi_{\mathrm{pot}} : \mathsf{J}_1\pi \to \mathsf{T}^*\mathsf{Q}$ such that $\mathfrak{D}_{\mathsf{d}}(V)(q) = \Phi_{\mathrm{pot}}(j_1 V(q))$. We denote by

$$\pi_{\mathcal{W}_{\mathrm{cl}}} : \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}/\mathcal{W}_{\mathrm{cl}}$$

the canonical projection.

DEFINITION 3.1. *Let $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathcal{F}_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an open-loop simple mechanical control system. The submanifold $\mathsf{R}_{\mathrm{pot}} \subset \mathsf{J}_1\pi$ defined by*

$$\mathsf{R}_{\mathrm{pot}} = \{p \in \mathsf{J}_1\pi \mid \pi_{\mathcal{W}_{\mathrm{cl}}} \circ \Phi_{\mathrm{pot}}(p) = \pi_{\mathcal{W}_{\mathrm{cl}}} \circ \Lambda_{cl}^{-1} \mathsf{d}V_{\mathrm{ol}}\}$$

*is called the* potential energy shaping submanifold. One can easily observe that the "equation" representation of $\mathsf{R}_{\mathrm{pot}}$ is given by (2.2).

Let $\pi_1 : \mathsf{J}_1\pi \to \mathsf{Q}$ be the canonical projection. Lewis [28] gives a set of sufficient conditions under which the potential shaping problem has a solution. The proof follows from the integrability theory of partial differential equations; in particular, the potential energy shaping partial differential equation has an involutive symbol; see [23, 20, 18] for the definition of involutivity. We recall the definition of ($\mathbb{G}_{\mathrm{ol}}$-$\mathbb{G}_{\mathrm{cl}}$)-potential energy shaping feedback from [28].

DEFINITION 3.2. *A section $\mathcal{F}$ of $\mathcal{W}$ is called a* ($\mathbb{G}_{\mathrm{ol}}$-$\mathbb{G}_{\mathrm{cl}}$)-potential energy shaping feedback *if there exists a function $V_{\mathrm{cl}}$ on $\mathsf{Q}$ such that*

$$\mathcal{F}(q) = \Lambda_{\mathrm{cl}} \mathsf{d}V_{\mathrm{cl}} - \mathsf{d}V_{\mathrm{ol}}, \qquad q \in \mathsf{Q}.$$

The following theorem implies when one can construct a Taylor series solution to the potential energy shaping partial differential equation order by order.

THEOREM 3.3. *Let $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathcal{F}_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an analytic open-loop simple mechanical control system. Let $\mathbb{G}_{\mathrm{cl}}$ be a closed-loop analytic metric. Let $p_0 \in \mathsf{R}_{\mathrm{pot}}$*

*and let* $q_0 = \pi_1(p_0)$*. Assume that* $q_0$ *is a regular point for* $\mathcal{W}_{\rm ol}$ *and that* $\mathcal{W}_{\rm cl} = \Lambda_{\rm cl}^{-1}\mathcal{W}_{\rm ol}$ *is integrable in a neighborhood of* $q_0$*. Then the following statements are equivalent:*

1. *There exists a neighborhood* $U$ *of* $q_0$ *and an analytic* $(\mathbb{G}_{\rm ol}\text{-}\mathbb{G}_{\rm cl})$*-potential energy shaping feedback* $\mathcal{F} \in \Gamma^\omega(\mathcal{W})$ *defined on* $U$ *which satisfies*

$$\Phi_{\rm pot}(p_0) = \Lambda_{\rm cl}\mathsf{d}V(q_0) - \mathsf{d}V_{\rm ol}(q_0) + \Lambda_{\rm cl}^{-1}\mathsf{d}V_{\rm ol}(q_0)$$

   *for a solution* $V$ *to* $\mathsf{R}_{\rm pot}$*.*

2. *There exists a neighborhood* $U$ *of* $q_0$ *such that* $d(\Lambda_{\rm cl}^{-1}\mathsf{d}V_{\rm ol})(q) \in \mathsf{I}_2(\mathcal{W}_{\rm cl}|_q)$*, where we denote* $\mathsf{I}_2(\mathcal{W}_{\rm cl}|_q) = \mathsf{I}(\mathcal{W}_{\rm cl}|_q) \cap \Lambda_2(\mathsf{T}_q^*\mathsf{Q})$ *and the algebraic ideal* $\mathsf{I}(\mathcal{W}_{\rm cl}|_q)$ *of* $\Lambda(\mathsf{T}_q^*\mathsf{Q})$ *is generated by elements of the form* $\gamma \wedge \omega$ *with* $\gamma \in \mathcal{W}_{\rm cl}|_q$*.*

The theorem gives a set of compatibility conditions for the existence of a $(\mathbb{G}_{\rm ol}$-$\mathbb{G}_{\rm cl})$-potential energy shaping feedback. Moreover, one can give a full description of the set of achievable potential energy shaping feedbacks. Let $\alpha_{cl} = \Lambda_{\rm cl}^{-1}\mathsf{d}V_{\rm ol}$. Let us use a coordinate system $(q^1, \dots, q^n)$ on $U$ a neighborhood of $q_0$ such that

$$\mathcal{W}_{\rm cl}|_{q_0} = \mathsf{span}(dq^{m+1}, \dots, dq^n).$$

In these local coordinates we write the one form $\alpha_{cl}$ as $\alpha_{cl} = \alpha_j dq^j$ and compatibility conditions become

$$\frac{\partial \alpha_j}{\partial q^i} - \frac{\partial \alpha_i}{\partial q^j} = 0, \quad i, j \in \{1, \dots, m\}. \tag{3.1}$$

**3.2. Kinetic energy shaping (the $\boldsymbol{\lambda}$-method).** In the following, we recall the so-called $\lambda$-method *in the absence of gyroscopic forces*. The idea is to transform the kinetic energy shaping partial differential equations to an overdetermined linear partial differential equation, the so-called $\lambda$-equation [5, 13, 18].

THEOREM 3.4. *Let* $\Sigma_{\rm ol} = (\mathsf{Q}, \mathbb{G}_{\rm ol}, V_{\rm ol}, \mathcal{F}_{\rm ol}, \mathcal{W}_{\rm ol})$ *be an open-loop simple mechanical control system. Let* $P \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}\mathsf{Q})$ *be the* $\mathbb{G}_{\rm ol}$*-orthogonal projection as above. Let* $\mathbb{G}_{\rm cl} \in \Gamma^\omega(\mathsf{S}_2^+\mathsf{T}^*\mathsf{Q})$*. If* $\mathbb{G}_{\rm ol}^\flat = \mathbb{G}_{\rm cl}^\flat \circ \lambda$ *for* $\lambda \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}\mathsf{Q})$*, the following two conditions are equivalent:*

1. $P(\nabla_X^{\mathbb{G}_{\rm cl}} X - \nabla_X^{\mathbb{G}_{\rm ol}} X) = 0 \quad \forall X \in \Gamma^\omega(\mathsf{T}\mathsf{Q})$.
2. (a) $\nabla_Z^{\mathbb{G}_{\rm ol}}(\mathbb{G}_{\rm ol}\lambda)(PX, PY) = 0$ *and*
   (b) $\nabla_{\lambda PX}^{\mathbb{G}_{\rm ol}}\mathbb{G}_{\rm cl}(Z, Z) + 2\mathbb{G}_{\rm cl}(\nabla_Z^{\mathbb{G}_{\rm ol}}\lambda PX, Z) = 2\mathbb{G}_{\rm ol}(\nabla_Z^{\mathbb{G}_{\rm ol}} PX, Z)$,
   *where* $X, Y, Z \in \Gamma^\omega(\mathsf{T}\mathsf{Q})$.

For a complete version of the theorem and the proof, in the presence of gyroscopic forces, see [13, 18]. The set of $\lambda$-equations have been proved to be formally integrable under a surjectivity condition [18].

**3.3. An important corollary for systems with one degree of underactuaion.** For systems with one degree of underactuation the potential energy shaping partial differential equations is always formally integrable. The main idea of the proof is that (3.1) vanishes for $m = 1$; for details of the proof, see [18].

THEOREM 3.5. *If* $\Sigma_{\rm ol}$ *is a simple mechanical control system with one degree of underactuation, for each bundle automorphism that satisfies the* $\lambda$*-equation, there exists a closed-loop metric and a closed-loop potential function that satisfy the energy shaping partial differential equations.*

In the rest of this paper, we focus on stabilization of the closed-loop system. Basically, we seek a solution to the energy shaping partial differential equation for which the Hessian of the closed-loop potential function can be guaranteed to be positive-definite.

**4. Stabilization of systems with one degree of underactuation.** In this section, we wish to determine the stabilizing solutions to the energy shaping partial differential equations for systems with one degree of underactuation. Throughout this section, let $\mathsf{Q}$ be an $n$-dimensional analytic manifold and $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}, V_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an open-loop simple mechanical control system with one degree of underactuation. We denote the Hessian of a potential function $V$ at $q_0 \in \mathsf{Q}$ by $\mathsf{Hess}(V)(q_0) \in \mathsf{S}_2\mathsf{T}^*_{q_0}\mathsf{Q}$. In particular, we denote the Hessian of the open-loop potential function and the closed-loop potential function at the equilibrium point $q_0$ by $\mathsf{Hess}(V_{\mathrm{ol}})(q_0)$ and $\mathsf{Hess}(V_{\mathrm{cl}})(q_0)$, respectively. Since the compatibility conditions of Theorem 3.3 are always satisfied for systems with one degree of underactuation, Theorem 3.5, one can study the prolongation of the potential energy shaping partial differential equations instead of the original partial differential equations. Let $(q^1, \dots, q^n)$ be a local coordinate in a neighborhood $U$ of $q_0 \in \mathsf{Q}$ such that $\mathcal{W}_{\mathrm{ol}} = \mathrm{span}\{dq^2, \dots, dq^n\}$ and let $P$ be the projection of $\mathsf{T}^*\mathsf{Q}$ onto $\mathsf{span}\{dq^1\}$.

If we prolong the potential energy shaping partial differential equation and evaluate the result at the origin, noting that $\mathsf{d}V_{\mathrm{cl}}(q_0) = 0$, we have

$$P\big(\mathbb{G}^\flat(q_0)\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)d^2V_{\mathrm{cl}}(v)(q_0) - d^2V_{\mathrm{ol}}(v)(q_0)\big) = 0,$$

where $v \in \mathsf{T}_{q_0}\mathsf{Q}$, i.e.,

$$\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0) - \mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\mathrm{ol}})(q_0) = \mathbb{G}^\sharp(q_0)(u|_{q_0}), \tag{4.1}$$

where $u : \mathsf{TQ} \to \mathcal{W}_{\mathrm{ol}}$. If the system is linearly controllable, then one can design a control such that $\mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\mathrm{ol}})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$ is diagonalizable and positive-definite. It is important to note that this does not necessarily imply that there exist $\mathbb{G}_{\mathrm{cl}}$ and $V_{\mathrm{cl}}$ such that $\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ is positive-definite, since the kinetic energy shaping partial differential equation puts restrictions on the achievable closed-loop metrics. However, we will show that, for systems with one degree of underactuation, the space of solutions of the kinetic energy shaping partial differential equations is large enough so that $\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ can be made positive-definite. We do this in the following steps:

1. We first identify a simple class of solutions to the $\lambda$-equation using Proposition 4.1.
2. We show that this class of solutions is large enough to ensure that (4.1) holds with $\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ diagonalizable and positive-definite.

Let $U$ be a neighborhood of the equilibrium point $q_0 \in \mathsf{Q}$ and let $(q^1, \dots, q^n)$ be the local coordinates on $U$. In order to find the class of solutions mentioned in 1, we need to make some observations about the kinetic energy shaping partial differential equations for systems with one degree of underactuation. For these systems, the $\lambda$-equation in the adapted local coordinate is given by

$$\frac{\partial}{\partial q^k}(\mathbb{G}_{1i}\lambda^i_1) - 2\mathcal{S}^s_{k1}\mathbb{G}_{si}\lambda^i_1 = 0, \tag{4.2}$$

where $\mathcal{S}^i_{jk}$, for $i, j, k \in \{1, \dots, n\}$, are the Levi–Civita connection coefficients associated to $\mathbb{G}$ and $i, k, s \in \{1, \dots, n\}$. Suppose we are seeking solutions to the $\lambda$-equation that in local coordinates look like $\lambda(q) = \lambda^j_i dq^i \otimes \frac{\partial}{\partial q^j}$, where $\lambda^j_i \in \mathbb{R}$ and $q \in U$, i.e., $\lambda$ is constant. Then one can write (4.2) as follows:

(4.3)
$$\left(\frac{\partial \mathbb{G}_{11}}{\partial q^k} - 2\mathcal{S}^i_{k1}\mathbb{G}_{i1}\right)\lambda^1_1 + \left(\frac{\partial \mathbb{G}_{12}}{\partial q^k} - 2\mathcal{S}^i_{k1}\mathbb{G}_{i2}\right)\lambda^2_1 + \cdots + \left(\frac{\partial \mathbb{G}_{1n}}{\partial q^k} - 2\mathcal{S}^i_{k1}\mathbb{G}_{in}\right)\lambda^n_1 = 0.$$

Because $\mathcal{S}$ is the Levi–Civita connection for $\mathbb{G}$, the first term vanishes, leaving $\lambda_1^1$ arbitrary. One can rewrite (4.3) in the following fashion:

$$\sum_{i=1}^{n}\sum_{j=2}^{n}(\mathcal{S}_{kj}^{i}\mathbb{G}_{i1} - \mathcal{S}_{k1}^{i}\mathbb{G}_{ij})\lambda_1^j = 0, \tag{4.4}$$

where $k \in \{1,\dots,n\}$. Thus, if $\lambda_2^j = 0$ for $j \in \{2,\dots,n\}$, $\lambda(q)$ is a solution to the $\lambda$-equation. Note that we further require that $\lambda(q) \circ \mathbb{G}^\sharp(q)$ is symmetric. In the following, we describe the space of such solutions of the $\lambda$-equation in an algebraic fashion.

Let $\mathsf{V}$ be an $n$-dimensional $\mathbb{R}$-vector space, and let $\mathsf{G} \in \mathsf{S}_2\mathsf{V}$ be a nondegenerate symmetric tensor. Let $\Phi_\mathsf{G} : \mathsf{V}^* \otimes \mathsf{V} \to \Lambda_2\mathsf{V}$ be the map defined by

$$\Phi_\mathsf{G}(\mathsf{A})(v_1, v_2) = \mathsf{A} \circ \mathsf{G}(v_1, v_2) - \mathsf{A} \circ \mathsf{G}(v_2, v_1),$$

where $v_1, v_2 \in \mathsf{V}$. The space of all tensors, $\mathsf{A} \in \mathsf{V}^* \otimes \mathsf{V}$, such that $\mathsf{A} \circ \mathsf{G}$ is symmetric, belongs to the kernel of $\Phi_\mathsf{G}$ and thus is of dimension $\frac{n(n+1)}{2}$; we denote this subspace by $\mathsf{S}_\mathsf{G}$. Let $\{e_i\}_{i=1}^n$ be a basis for $\mathsf{V}$ and let $\{e^i\}_{i=1}^n$ be its dual. Let $\mathsf{W} \subset \mathsf{V}^*$ be the vector subspace generated by $\{e^2,\dots,e^n\}$ and denote its complement by $\mathsf{E}$. We denote by $\tilde{\mathsf{S}}$ the space of all $\mathsf{A} \in \mathsf{V}^* \otimes \mathsf{V}$ such that if $v \in \mathrm{coann}(\mathsf{W})$, then $\mathsf{A}(v) \in \mathrm{coann}(\mathsf{W})$ for all $v \in \mathsf{V}$. A tensor $\mathsf{A} \in \tilde{\mathsf{S}}$ can be written as

$$\mathsf{A} = \mathsf{A}_1^1 e^1 \otimes e_1 + \sum_{i=2}^{n}\sum_{j=1}^{n} \mathsf{A}_i^j e^i \otimes e_j,$$

where $\mathsf{A}_1^1 \in \mathbb{R}$ and $\mathsf{A}_i^j \in \mathbb{R}$ for $i \in \{2,\dots,n\}$ and $j \in \{1,\dots,n\}$. Thus the dimension of $\tilde{\mathsf{S}}$ is $n(n-1)+1$. If we denote the restriction of the map $\Phi_\mathsf{G}$ to $\tilde{\mathsf{S}}$ by $\Phi_\mathsf{G}|_{\tilde{\mathsf{S}}} : \tilde{\mathsf{S}} \to \Lambda_2\mathsf{V}$, then $\ker(\Phi_\mathsf{G}|_{\tilde{\mathsf{S}}})$ is of dimension $\frac{n(n-1)}{2}+1$. If we additionally require that $\mathsf{A} \in \ker(\Phi_\mathsf{G}|_{\tilde{\mathsf{S}}})$ be nondegenerate, we obtain a $\frac{n(n-1)}{2}$-dimensional subspace of $\mathsf{V}^* \otimes \mathsf{V}$.

Let $\mathsf{Q}$ be an $n$-dimensional analytic manifold and let $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}, V_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an open-loop simple mechanical control system with one degree of underactuation. Let $U$ be a neighborhood of the equilibrium point $q_0 \in \mathsf{Q}$ and let $(q^1,\dots,q^n)$ be local coordinates on $U$ such that $\mathcal{W}_{\mathrm{ol}}|_q = \mathsf{span}\{dq^2,\dots,dq^n\}$, where $q \in U$. In the following, we define a subspace of $\mathsf{T}_q^*\mathsf{Q} \otimes \mathsf{T}_q\mathsf{Q}$, which is large enough for the stabilization of systems with one degree of underactuation. Consider the space of solutions to the $\lambda$-equation that in local coordinates look like $\lambda(q) = \lambda_i^j dq^i \otimes \frac{\partial}{\partial q^j} \in \mathsf{T}_q^*\mathsf{Q} \otimes \mathsf{T}_q\mathsf{Q}$, where $\lambda_i^j \in \mathbb{R}$ and $q \in U$, and satisfy the following:

1. $\lambda(q) \circ \mathbb{G}^\sharp(q)$ is symmetric and nondegenerate.
2. If $v \in \mathrm{coann}(\mathsf{span}\{dq^1\})$, then $\lambda(v) \in \mathrm{coann}(\mathsf{span}\{dq^1\})$ for all $v \in \mathsf{T}_q\mathsf{Q}$.

We denote this subspace by $\mathscr{S}$. The following proposition is a corollary of the algebraic discussion above.

PROPOSITION 4.1. *$\mathscr{S}$ is an $\frac{n(n-1)}{2}$-dimensional subspace of $\mathsf{T}_q^*\mathsf{Q} \otimes \mathsf{T}_q\mathsf{Q}$.*

We wish to show that the space of solutions of the $\lambda$-equation, described in Proposition 4.1, is large enough to guarantee that $\mathbb{G}_{\mathrm{cl}}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ can be made diagonalizable and with positive real eigenvalues. If $\lambda(q) \in \mathscr{S}$, then (4.1) gives

$$\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)\left(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j}\right) = \frac{1}{\lambda_1^1}\mathsf{Hess}^\flat(V_{\mathrm{ol}})(q_0)\left(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j}\right), \tag{4.5}$$

$$\mathbb{G}_{\mathrm{cl}}^\sharp(q_0)(dq^1, dq^j) = \lambda_1^1\mathbb{G}^\sharp(q_0)(dq^1, dq^j), \tag{4.6}$$

where $j \in \{1,\dots,n\}$. As a result, we have the following proposition.

PROPOSITION 4.2. *Let* $\mathsf{Q}$ *be an* $n$*-dimensional analytic manifold and let* $\Sigma_{\rm ol} = (\mathsf{Q}, \mathbb{G}, V_{\rm ol}, \mathcal{W}_{\rm ol})$ *be an open-loop simple mechanical control system with one degree of underactuation. Let* $U$ *be a neighborhood of the equilibrium point* $q_0 \in \mathsf{Q}$ *and let* $(q^1, \ldots, q^n)$ *be local coordinates on* $U$ *such that* $\mathcal{W}_{\rm ol}|_q = \mathsf{span}\{dq^2, \ldots, dq^n\}$, *where* $q \in U$. *Suppose that*

$$A = \mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\rm ol})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$$

*is diagonalizable with real eigenvalues, where* $u|_{q_0} : \mathsf{T}_{q_0}\mathsf{Q} \to \mathcal{W}_{\rm ol}|_{q_0}$. *Then there exist a closed-loop metric* $\mathbb{G}_{\rm cl}$ *and a potential function* $V_{\rm cl}$ *such that*

1. $\mathbb{G}^\flat = \mathbb{G}^\flat_{\rm cl} \circ \lambda$, *where* $\lambda \in \mathscr{S}$,
2. $\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}^\flat(V_{\rm cl})(q_0) = A$.

*Proof.* We need only show that if 1 above holds, then $\mathbb{G}_{\rm cl}$ and $V_{\rm cl}$ can be selected so that 2 above holds. Using (4.5) and (4.6), we can write $\mathbb{G}^\sharp_{\rm cl}(q_0)$ in coordinates as

$$\begin{pmatrix} \lambda^1_1 a & \lambda^1_1 \mathsf{B} \\ \lambda^1_1 \mathsf{B}^T & \mathsf{C} \end{pmatrix},$$

where $a \in \mathbb{R}$, $\mathsf{B} \in \mathrm{L}(\mathbb{R}^{n-1}, \mathbb{R})$, and $\mathsf{C} \in \mathsf{S}_2\mathbb{R}^{n-1}$ are such that $a = \mathbb{G}^\sharp(dq^1, dq^1)$ and $\mathsf{B}(dq^1, dq^j) = \mathbb{G}^\sharp(dq^1, dq^j)$ for all $j \in \{2, \ldots, n\}$. Similarly, $\mathsf{Hess}^\flat(V_{\rm cl})(q_0)$ can be written as

$$\begin{pmatrix} \frac{1}{\lambda^1_1} k & \frac{1}{\lambda^1_1}\mathcal{B} \\ \frac{1}{\lambda^1_1}\mathcal{B}^T & \mathcal{C} \end{pmatrix},$$

where $k \in \mathbb{R}$, $\mathcal{B} \in \mathrm{L}(\mathbb{R}^{n-1}, \mathbb{R})$, and $\mathcal{C} \in \mathsf{S}_2\mathbb{R}^{n-1}$ are such that $k = \mathsf{Hess}^\flat(V_{\rm ol})(q_0)(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^1})$ and $\mathcal{B}(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j}) = \mathsf{Hess}^\flat(V_{\rm ol})(q_0)(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j})$ for all $j \in \{2, \ldots, n\}$. Thus we have

$$\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}^\flat(V_{\rm cl})(q_0) = \mathbb{G}^\sharp_{\rm ol}(q_0)\mathsf{Hess}^\flat(V_{0l})(q_0) + \begin{pmatrix} 0 & 0 \\ L_1 & L_2 \end{pmatrix},$$

where

1. $L_1 = k\mathsf{B}^T + \frac{1}{\lambda^1_1}\mathsf{C}\mathcal{B}^T \in \mathrm{L}(\mathbb{R}, \mathbb{R}^{n-1})$ and
2. $L_2 = \lambda^1_1 \mathsf{B}\mathcal{B}^T + \mathsf{C}\mathcal{C}^T \in \mathrm{L}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$

can be set to any value by appropriate choice of $\mathsf{C}$ and $\mathcal{C}$. ∎

THEOREM 4.3. *Let* $\Sigma_{\rm ol} = (\mathsf{Q}, \mathbb{G}_{\rm ol}, V_{\rm ol}, \mathcal{W}_{\rm ol})$ *be a linearly controllable open-loop simple mechanical control system with one degree of underactuation and with* $q_0 \in \mathsf{Q}$ *an equilibrium point. Then the system is stabilizable at* $q_0$ *using an energy shaping feedback.*

*Proof.* The integrability of the energy shaping partial differential equations ensures that formal solutions exist. Furthermore, Theorem 3.5 implies that the obstructions of Theorem 3.3 are satisfied for systems with one degree of underactuation. If the system is linearly controllable, then one can design a control such that $\mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\rm ol})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$ is diagonalizable and positive-definite. Proposition 4.2 then guarantees that $\mathbb{G}_{\rm cl}$ can be found such that it satisfies the kinetic energy shaping partial differential equations, by choosing $\lambda \in \mathscr{S}$ and taking

$$\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}^\flat(V_{\rm cl})(q_0) = \mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\rm ol})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$$

to be diagonalizable with positive real eigenvalues. ∎

Note that this proof does not require that the closed-loop metric be positive-definite and, in fact, in the absence of gyroscopic forces, there are cases for which energy shaping is not possible with positive-definite closed-loop metrics; an example of this is presented in Example 5. The following proposition clarifies when it is necessary to perform kinetic energy shaping for systems with one degree of underactuation.

PROPOSITION 4.4. *Let* $\mathsf{Q}$ *be an* $n$*-dimensional manifold and let* $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}, V_{\text{ol}}, \mathcal{W}_{\text{ol}})$ *be a linearly controllable simple mechanical system. Let* $U$ *be a neighborhood of* $q_0 \in \mathsf{Q}$ *such that* $\mathcal{W}_{\text{ol}} = \mathsf{span}\{dq^2, \dots, dq^n\}$*. If* $\mathsf{Hess}(V_{\text{ol}})(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^1}) > 0$*, the system can be stabilized around its equilibrium point* $q_0$ *without kinetic energy shaping.*

*Proof.* We shall show that $\Sigma_{\text{ol}}$ is stabilizable using an energy shaping feedback with $\mathbb{G}_{\text{cl}} = \mathbb{G}$. Equation (4.1) then reads

$$\mathsf{Hess}^\flat(V_{\text{cl}})(q_0) = \mathsf{Hess}^\flat(V_{\text{ol}})(q_0) + u|_{q_0},$$

where $u$ is a feedback. Note that since $\mathsf{Hess}(V_{\text{cl}})$ is symmetric, it is positive-definite if and only if all of its principal minors are positive. The first principal minor of $\mathsf{Hess}^\flat(V_{\text{cl}})$ is positive. Then, by linear controllability, one can choose the controls so that the system is stabilizable at the equilibrium point $q_0$, similar to Proposition 4.2. ∎

Next, we present an example of energy shaping for simple mechanical systems with one degree of underactuation for which the energy shaping is possible *only* via a closed-loop metric that is not positive-definite.

**5. Example.** Consider the stabilization problem for a simple mechanical control system $\Sigma = (\mathbb{R}^2, \mathbb{G}, V_{\text{ol}}, 0, \mathsf{W}_{\text{ol}})$ at the origin $q_0 = \mathbf{0} \in \mathbb{R}^2$, where

1. $\mathbb{G} = ((q^2)^2 + 1)dq^1 \otimes dq^2 + ((q^1)^2 + 1)dq^2 \otimes dq^2$,
2. $V_{\text{ol}} = -(q^1)^2 + 2q^1q^2 + (q^2)^2$, and
3. $\mathcal{W}_{\text{ol}} = \mathsf{span}\{dq^2\}$.

This system is linearly controllable at the origin. We show that, for any solution of the $\lambda$-equation, the constant term in the Taylor expansion of $\lambda_1^2$ is always zero. In order to show this, we need to modify (4.4) by adding an extra term, since $\lambda$, in a neighborhood of $q_0$, is not necessarily chosen from $\mathscr{S}$. We have

$$\sum_{i=1}^{n} \left( \mathbb{G}_{1i} \frac{\partial \lambda_1^i}{\partial q^k} + \sum_{j=2}^{n} (\mathcal{S}_{kj}^i \mathbb{G}_{i1} - \mathcal{S}_{k1}^i \mathbb{G}_{ij}) \lambda_1^j \right) = 0$$

for all $k \in \{1, \dots, n\}$. For this example, by substituting the nonzero Christoffel symbols, we have

$$((q^2)^2 + 1)\frac{\partial \lambda_1^1}{\partial q^1} + 2q^2\lambda_1^2 = 0, \tag{5.1}$$

$$((q^2)^2 + 1)\frac{\partial \lambda_1^1}{\partial q^2} - 2q^1\lambda_1^2 = 0. \tag{5.2}$$

It is clear that $\lambda_1^1(q_0)$ can be chosen arbitrarily. Consider formal expressions for $\lambda_1^2$ and $\lambda_1^1$:

$$\lambda_1^1 = C_{00} + C_{10}q^1 + C_{01}q^2 + C_{20}(q^1)^2 + C_{02}(q^2)^2 + C_{11}q^1q^2 + \cdots,$$
$$\lambda_1^2 = D_{00} + D_{10}q^1 + D_{01}q^2 + D_{20}(q^1)^2 + D_{02}(q^2)^2 + D_{11}q^1q^2 + \cdots,$$

where $C_{ij}, D_{ij} \in \mathbb{R}$ for $i, j \in \mathbb{Z}_{\geq 0}$. If $\lambda_1^1$ and $\lambda_1^2$ satisfy (5.1) and (5.2), then $C_{11} = D_{00} = 0$, i.e., $\lambda_1^2(q_0) = 0$. Thus the closed-loop metric at the origin has the form $\mathbb{G}_{\mathrm{cl}}(q_0) = \frac{1}{a} dq^1 \otimes dq^1 + \frac{1}{c} dq^2 \otimes dq^2$, where $a, c \in \mathbb{R}\backslash\{0\}$ and $\lambda_1^1(q_0) = a$. Equation (4.5) implies that

$$\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0) = \begin{pmatrix} \frac{-2}{a} & \frac{2}{a} \\ \frac{2}{a} & k \end{pmatrix},$$

where $k \in \mathbb{R}$. Thus

$$\mathbb{G}_{\mathrm{cl}}^\sharp(q_0) \mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0) = \begin{pmatrix} -2 & 2 \\ \frac{2c}{a} & ck \end{pmatrix}.$$

It is easy to see that one has to choose $\frac{2c}{a} < 0$ and $ck > 2$ in order to make $\mathbb{G}_{\mathrm{cl}}^\sharp(q_0) \mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ positive-definite; i.e., none of the achievable closed-loop metrics is positive-definite. However, one can choose $a, c, k \in \mathbb{R}$ so that $\mathbb{G}_{\mathrm{cl}}^\sharp(q_0) \mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ is positive-definite, for example, $a = -\frac{191}{100}$, $c = \frac{43}{10}$, and $k = 1$.

*Remark.* If we take the open-loop metric given by

$$\mathbb{G} = ((q^2)^2 + 1) dq^1 \otimes dq^2 + ((q^1)^2 + 1) dq^2 \otimes dq^2 + 2q^1 q^2 (dq^1 \otimes dq^2 + dq^2 \otimes dq^1),$$

then $\lambda_1^2(q_0)$ need not be zero and the system can be shown to be stabilizable by the energy shaping method with a positive-definite closed-loop metric. This reveals that a slight change in the structure of the open-loop Levi–Civita connection has a huge impact on the achievable closed-loop metrics.

**6. Summary.** In this paper, we fully characterized the problem of stabilization of systems with one degree of underactuation using energy shaping method. The results completely relied on the integrability analysis of partial differential equations involved in energy shaping. We illustrated that all linearly controllable simple mechanical control systems with one degree of underactuation can be stabilized using an energy shaping feedback, with closed-loop metrics which are not necessarily positive-definite. We also characterized the simple mechanical systems for which the energy shaping is achievable without kinetic energy shaping. Finally, we gave an example of a simple mechanical control system with one degree of underactuation for which, in the absence of gyroscopic forces, there exists no solution to the energy shaping problem with positive-definite closed-loop metric. The results give some useful insight about the structure of kinetic energy shaping.

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