Asymptotic nature of higher Mahler measure

by

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1. Introduction

DEFINITION 1.1. Given a nonzero Laurent polynomial $P(x) \in \mathbb{C}[x^{\pm 1}]$ and $k \in \mathbb{N}$, the $k$-higher Mahler measure of $P$ (see [4]) is defined by

$$m_k(P) := \int_0^1 \log^k |P(e^{2\pi i \theta})| \, d\theta = \frac{1}{2\pi i} \int_{|z|=1} \log^k |P(z)| \frac{dz}{z}.$$ 

These $m_k$'s are multiples of the coefficients in the Taylor expansion of Akatsuka’s zeta Mahler measure (see [2])

$$Z(s, P) := \int_0^1 |P(e^{2\pi i \theta})|^s \, d\theta,$$

that is,

$$Z(s, P) = \sum_{k=0}^{\infty} \frac{m_k(P)}{k!} s^k.$$

For $k = 0, 1, 2, \ldots$, let $a_k(P) = m_k(P)/k!$, so that

$$Z(s, P) = \sum_{k=0}^{\infty} a_k(P) s^k.$$

In this paper we only consider polynomials of type $P(x) = x - r$ with $|r| = 1$. Therefore, from now on, we write $m_k(x - r) = m_k$ and $a_k(x - r) = a_k$ for simplicity.

2. Asymptotic nature of higher Mahler measure of $r - x$ when $|r| = 1$. We will prove

THEOREM 2.1. Let $m_k$ and $a_k$ be as above. Then

(a) $\frac{m_{k+1}}{(k+1)!} + \frac{m_k}{k!} = a_{k+1} + a_k = O(1/k)$,

(b) $\lim_{k \to \infty} \left| \frac{m_k}{k!} \right| = \lim_{k \to \infty} |a_k| = \frac{1}{\pi}$.

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(c) \( \frac{m_{k+1}}{(k+1)!} + \frac{m_k}{k!} = a_{k+1} + a_k = o(1/k), \)

(d) \( \lim_{k \to \infty} \frac{1}{k+1} \cdot \frac{m_{k+1}}{m_k} = \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = -1. \)

From [4] we know that for \(|s| < 1,\)

\[
Z(s, r - x) = \exp \left( \sum_{k=2}^{\infty} \frac{(-1)^k (1 - 2^{1-k}) \zeta(k) s^k}{k} \right).
\]

Differentiating both sides of (2.1) with respect to \(s\) we obtain

\[
\sum_{k=1}^{\infty} k a_k s^{k-1} = \frac{\partial}{\partial s} Z(s, r - x) = Z(s, r - x) \sum_{k=2}^{\infty} (-1)^k (1 - 2^{1-k}) \zeta(k) s^{k-1}
\]

\[
= \left( \sum_{k=0}^{\infty} a_k s^k \right) \left( \sum_{k=1}^{\infty} b_k s^k \right) = \sum_{k=1}^{\infty} \left( a_0 b_k + \sum_{j=1}^{k-1} a_j b_{k-j} \right) s^k,
\]

where \(b_{k-1} := (-1)^k (1 - 2^{1-k}) \zeta(k).\) From the power series expansion of (2.1) we already know that \(a_0 = 1.\) Now comparing coefficients on both sides of the last expression we get \(a_1 = 0, a_2 = \frac{1}{2} a_0 b_1 = \frac{1}{4} \zeta(2)\) and for \(k \geq 3,\)

\[
a_k = \frac{1}{k} \sum_{j=0}^{k-2} a_j b_{k-1-j},
\]

where

\[
b_k := (-1)^{k+1} (1 - 2^{-k}) \zeta(k + 1).
\]

3. A few remarks and lemmas

**Remark 3.1.** It can be easily shown by induction that \(a_{2k} > 0\) and \(a_{2k+1} < 0\) for all \(k \geq 1.\) It is also easy to see that

\[
a_k = \frac{(-1)^k}{k} \sum_{j=0}^{k-2} |a_j b_{k-1-j}| \quad \text{for } k > 1.
\]

**Remark 3.2.** Let \(B_k := |b_k|\). Then \(B_k \leq 1\) for all \(k \geq 1, B_k\) is increasing and \(B_k \to 1\) as \(k \to \infty.\)

Notice \(B_k = \eta(k+1)\) where \(\eta(k)\) is Dirichlet’s eta function. Since \(\eta(k) \to 1\) as \(k \to \infty\) and \(\eta(k)\) is an increasing function of \(k\) by \([1], B(k) \leq 1\) for all \(k \geq 1, B_k\) is increasing and \(B_k \to 1\) as \(k \to \infty.\)

**Lemma 3.3.** \(|a_k| \leq 1\) for all \(k \geq 1.\)
**Proof.** We use induction. First we see that $|a_0| = 1 \leq 1$, $|a_1| = 0 \leq 1$, and $|a_2| = \zeta(2)/4 = \pi^2/24 \leq 1$. Now assume $|a_j| \leq 1$ for all $2 < j < k$. Using this along with Remark 3.2, we get

$$|a_k| = \left| \frac{1}{k} \sum_{j=0}^{k-2} a_j b_{k-1-j} \right| \leq \frac{1}{k} \sum_{j=0}^{k-2} |a_j b_{k-1-j}| \leq \frac{1}{k} \sum_{j=0}^{k-2} 1 = \frac{k-1}{k} < 1.$$ 

**Lemma 3.4.** For $k \geq 4$, $\zeta(k) - \zeta(k+1) \leq 1/k^2$.

**Proof.** We use induction. First notice that for all $k \geq 4$ and $n \geq 2$ we have $0 < \sqrt{n}/(\sqrt{n} - 1) < 4 \leq k$, from which it follows that $(n - 1/k)^2 \geq 1$. For $k = 4$ we see that $\zeta(4) - \zeta(5) \approx 0.045 < 0.0625 = 1/4^2$. Assume the conclusion of the lemma is true for all $4 < j < k$, in particular for $j = k-1$. Since for all $k \geq 4$ and $n \geq 2$ we have $n(1 - 1/k)^2 \geq 1$, it follows that

$$\frac{1}{k^2} = \left( \frac{k-1}{k} \right)^2 \cdot \frac{1}{(k-1)^2} \geq \left( 1 - \frac{1}{k} \right)^2 (\zeta(k-1) - \zeta(k))$$

$$= \sum_{n=2}^{\infty} n \left( 1 - \frac{1}{k} \right)^2 \left( \frac{1}{n^k} - \frac{1}{n^{k+1}} \right)$$

$$\geq \sum_{n=2}^{\infty} \left( \frac{1}{n^k} - \frac{1}{n^{k+1}} \right) = \zeta(k) - \zeta(k+1).$$

**Lemma 3.5.** Recall $B_k = |b_k|$. For $k > 1$,

$$B_k - B_{k-1} \leq 1/k^2.$$

**Proof.** Indeed,

$$\frac{1}{k^2} - (B_k - B_{k-1}) = \frac{1}{k^2} - B_k + B_{k-1}$$

$$= \frac{1}{k^2} - \left( 1 - \frac{1}{2^k} \right) \zeta(k+1) + \left( 1 - \frac{1}{2^{k-1}} \right) \zeta(k)$$

$$= \frac{1}{k^2} - \left( 1 - \frac{1}{2^k+1} + \frac{1}{3^k+1} - \frac{1}{4^k+1} + \cdots \right) + \left( 1 - \frac{1}{2^k+1} + \frac{1}{3^k} - \frac{1}{4^k} + \cdots \right)$$

$$= \frac{1}{k^2} - \frac{1}{2^k} \left( 1 - \frac{1}{2} \right) + \frac{1}{3^k} \left( 1 - \frac{1}{3} \right) - \frac{1}{4^k} \left( 1 - \frac{1}{4} \right) + \cdots$$

$$> \frac{1}{k^2} - \frac{1}{2^k} \left( 1 - \frac{1}{2} \right) > 0 \quad \text{for all } k > 1.$$ 

4. **Proofs of the results of Section 2**

**Proof of Theorem 2.1(a).** Using (2.3) and Lemma 3.4, notice that for $k - j \geq 4$,
\[
\left| \frac{b_{k-j} + b_{k-1-j}}{k} \right| \\
= \left| \frac{(-1)^{k-j+1}(1 - 2^{-k-j})\zeta(k - j + 1) + (-1)^{k-j}(1 - 2^{-k+1+j})\zeta(k - j)}{k} \right| \\
= \left| \frac{(1 - 2^{-k+1+j})\zeta(k - j) - (1 - 2^{-k+j})\zeta(k - j + 1)}{k} \right| \\
= \frac{1}{k(k+1)} \left| (k+1) \left( 1 - \frac{1}{2^{k-1-j}} \right)\zeta(k - j) - k \left( 1 - \frac{1}{2^{k-j}} \right)\zeta(k - j + 1) \right| \\
= \frac{1}{k(k+1)} \left| k(\zeta(k - j) - \zeta(k - j + 1)) - \frac{k}{2^{k-j}}(2\zeta(k - j) - \zeta(k - j + 1)) \right| \\
\leq \frac{1}{k(k+1)} \left[ k(\zeta(k - j) - \zeta(k - j + 1)) + \frac{k}{2^{k-j}} \{(\zeta(k - j) - \zeta(k - j + 1)) + \zeta(k - j)\} \right] \\
+ \frac{k}{2^{k-j}} \left\{ (\zeta(k - j) - \zeta(k - j + 1)) + \zeta(k - j) \right\} + \left( 1 - \frac{1}{2^{k-1-j}} \right)\zeta(k - j) \\
\leq \frac{1}{k(k+1)} \left[ \frac{k}{(k-j)^2} + \frac{k}{2^{k-j}} \left\{ \frac{1}{(k-j)^2} + \zeta(2) \right\} \right] \\
= \frac{1}{(k+1)(k-j)^2} + \frac{1}{2^{k-j}(k+1)(k-j)^2} + \frac{\zeta(2)}{2^{k-j}(k+1)} + \frac{\zeta(2)}{k(k+1)} \\
\leq \frac{2}{(k+1)(k-j)^2} + \frac{\zeta(2)}{2^{k-j}(k+1)} + \frac{\zeta(2)}{k(k+1)}.
\]

Therefore,
\[
|a_{k+1} + a_k| \\
= \left| \frac{1}{k+1} \sum_{j=0}^{k-1} a_j b_{k-j} + \frac{1}{k} \sum_{j=0}^{k-2} a_j b_{k-1-j} \right| \\
= \left| \frac{a_{k-1} b_1}{k+1} + \sum_{j=0}^{k-2} a_j \left( \frac{b_{k-j}}{k+1} + \frac{b_{k-1-j}}{k} \right) \right| \\
\leq \frac{1}{k+1} + \sum_{j=0}^{k-2} \left| \frac{b_{k-j}}{k+1} + \frac{b_{k-1-j}}{k} \right| \quad \text{by Remark (3.2) and Lemma (3.3)} \\
\leq \frac{1}{k+1} + \sum_{j=0}^{k-4} \left| \frac{b_{k-j}}{k+1} + \frac{b_{k-1-j}}{k} \right| + 2 \cdot \max\{|b_3|, |b_2|\} + 2 \cdot \max\{|b_2|, |b_1|\}.
\]
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\[
\begin{align*}
&\leq \frac{1}{k+1} + \sum_{j=0}^{k-4} \left[ \frac{2}{(k+1)} \cdot \frac{1}{(k-j)^2} + \frac{\zeta(2)}{(k+1)} \cdot \frac{1}{2^{k-j}} + \frac{\zeta(2)}{k(k+1)} \right] + \frac{4}{k} \\
&\leq \frac{5}{k} + \frac{2}{k+1} \sum_{j=0}^{k-4} \frac{1}{(k-j)^2} + \frac{\zeta(2)}{k+1} \sum_{j=0}^{k-4} \frac{1}{2^{k-j}} + \frac{\zeta(2)}{k(k+1)} \sum_{j=0}^{k-4} 1 \\
&= \frac{5}{k} + \frac{2}{k+1} \left( \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{k^2} \right) + \frac{\zeta(2)}{k+1} \left( \frac{1}{2^2} + \frac{1}{2^5} + \cdots + \frac{1}{2^k} \right) \\
&\quad + \frac{\zeta(2)(k-3)}{k(k+1)} \\
&\leq \frac{5}{k} + \frac{2}{k+1} \cdot \zeta(2) + \frac{\zeta(2)}{k+1} \cdot \frac{1}{1-1/2} + \frac{\zeta(2)}{k+1} \\
&= \frac{5}{k} + \frac{5\zeta(2)}{k+1} \leq \frac{5}{k} (1 + \zeta(2)).
\end{align*}
\]

Therefore for \( k \geq 4 \),

\[ |a_{k+1} + a_k| \leq \frac{5}{k} (1 + \zeta(2)), \]

and so \( a_{k+1} + a_k = O(1/k) \). ■

Proof of Theorem 2.1(b). By definition of the Akatsuka zeta Mahler measure (see [2]), the generating function \( f(s) \) of \( a_k \)'s is precisely \( Z(s, x-r) \) with \( |r| = 1 \). From [4] we know that for \( |r| = 1 \) and \( |s| < 1 \),

\[ f(s) := \sum_{k=0}^{\infty} a_k s^k = Z(s, x-r) = \frac{\Gamma(s+1)}{\Gamma^2(s/2+1)} = \frac{4}{s} \frac{\Gamma(s)}{\Gamma^2(s/2)}. \]

Define

\[ F(s) := 1 + \sum_{k=1}^{\infty} (-1)^k (a_{k-1} + a_k) s^k. \]

So, \( F(s) = (1-s)f(-s) \). Notice that

\[ \lim_{s \to 1^-} F(s) = -\frac{4}{\Gamma^2(-1/2)} \quad \lim_{s \to 1^-} (1-s)\Gamma(-s) = -\frac{4}{\Gamma^2(-1/2)} \quad \lim_{s \to -1} (1+s)\Gamma(s) = \frac{1}{\pi}, \]

since \( \lim_{s \to -1} (1+s)\Gamma(s) = -1 \) and \( \sqrt{\pi} = \Gamma(1/2) = (-1/2)\Gamma(-1/2) \).

Now \( \{k(-1)^k(a_k + a_{k+1})\} \) is a bounded sequence by Theorem 2.1(a). Therefore applying Littlewood’s extension of Tauber’s Theorem (see [3]) to the sequence \( \{(-1)^k(a_k + a_{k+1})\} \) and its generating function \( F(s) - 1 \) we see that

\[ \lim_{k \to \infty} |a_k| = 1 - \sum_{k=0}^{\infty} \{(-1)^k(a_k + a_{k+1})\} = 1 + \lim_{s \to 1^-} (F(s) - 1) = \frac{1}{\pi}. \]
Proof of Theorem 2.1(c). Recall $B_k = |b_k|$ from Lemma 3.5. Now define a new sequence $\{A_k\}$ by setting $A_0 = 1$, $A_1 = 0$ and

$$A_k = \frac{1}{k} \sum_{j=0}^{k-2} A_j B_{k-1-j}$$

for all $k \geq 2$. A careful observation of the individual terms inside $a_k$ and $A_k$ easily shows that $A_k = |a_k|$. Clearly $A_k = |a_k| \leq 1$ by Lemma 3.3. Let $m := \lfloor (k-2)/2 \rfloor$ and $A := 1/\pi$. Since $\lim_{k \to \infty} A_k = 1/\pi = A$, using Remark 3.2 and Lemma 3.5 we see that for each $\epsilon > 0$ there is a sufficiently large integer $N > 0$ such that $k > N$ implies

$$|(k+1)(a_{k+1} + a_k)| = |(k+1)(A_{k+1} - A_k)|$$

$$= \left| \sum_{j=0}^{k-1} A_j B_{k-j} - \sum_{j=0}^{k-2} A_j B_{k-1-j} - A_k \right|$$

$$\leq \left| A_{k-1} B_1 - A_k + \sum_{j=m+1}^{k-2} A_j (B_{k-j} - B_{k-1-j}) \right|$$

$$+ \sum_{j=0}^{m} A_j (B_{k-j} - B_{k-1-j}). \tag{4.1}$$

Now if the term within the absolute value signs in (4.1) is positive, then

$$|(k+1)(a_{k+1} + a_k)|$$

$$\leq \left| (A + \epsilon) B_1 - (A - \epsilon) + (A + \epsilon) \sum_{j=m+1}^{k-2} (B_{k-j} - B_{k-1-j}) \right|$$

$$+ \sum_{j=0}^{m} \frac{A_j}{(k-j)^2}$$

$$\leq \left| (A + \epsilon) B_1 - (A - \epsilon) + (A + \epsilon)(B_{k-m-1} - B_1) \right|$$

$$+ \frac{1}{(k-m)^2} (m+1)$$

Notice that $B_{k-m-1} \to 1$ and $(m+1)/(k-m)^2 \to 0$ as $k \to \infty$. Therefore

$$\lim_{k \to \infty} |(k+1)(a_{k+1} + a_k)| \leq |(A + \epsilon) B_1 - (A - \epsilon) + (A + \epsilon)(1 - B_1)|.$$

Since this inequality holds for each fixed $\epsilon > 0$, it also holds for $\epsilon = 0$. Hence $|(k+1)(a_{k+1} + a_k)| \to 0$ as $k \to \infty$. Therefore, $a_{k+1} + a_k = o(1/k)$.

If the term within the absolute value signs in (4.1) is negative, then a similar argument gives the same conclusion just by replacing $+\epsilon$ by $-\epsilon$ in (4.2). \qed
Proof of Theorem 2.1(d). From Theorem 2.1(b) we know that $0 < \lim_{k \to \infty} |a_k| = 1/\pi < \infty$. Now using Remark 3.1 we have
\[
\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = -1.
\]

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