RESEARCH STATEMENT OF ARUNABHA BISWAS

My research interests lie in analytic number theory and special functions. More specifically, I am developing a systematic study of the new field of “higher Mahler measure” and its connection with special values of $L$-functions.

$k$-higher Mahler measure ($m_k$) is a generalization of the classical (logarithmic) Mahler measure ($m$), i.e., $m = m_1$. For a Laurent polynomial $P(z) \in \mathbb{C}[z, z^{-1}]$, the logarithmic higher Mahler measure of order $k$ is defined as $m_k(P) := \int_0^1 \log^k |P(e^{2\pi it})| \, dt$ (by Kurokawa, Lalín, Ochiai [9] in 2008 and Akatsuka [1] in 2009). For $k = 1$, $m_1(P)$ is the classical logarithmic Mahler measure, which by Jensen’s formula [7], is given by $m(P) := \log (|a| \prod_{j=1}^{n} \max\{1, |r_j|\})$ for $P(z) = a \prod_{j=1}^{n} (z - r_j)$. In case the polynomial has integer coefficients, this quantity has arithmetic significance and appears in several seemingly unrelated problems. Mahler measure arose in 1933 from a pioneering work by Lehmer [11] while constructing large primes in a way that generalized the Mersenne primes. He conjectured (still open) that the polynomial $z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$ has the least non-zero Mahler measure among all polynomials with integer coefficients. The subject has attracted considerable attention in the last few decades since it has connections to areas beyond number theory, such as topological entropies of dynamical systems and polynomial knot invariants.

In my first project, I proved that for the simple polynomial $P(z) = z + r$ where $r$ is on the complex unit circle $S^1$, we have $|m_k(P)|/k! \to 1/\pi$ as $k \to \infty$. Using this, I also proved that for the same polynomial $P$, we have $m_k(P)/k! + m_{k+1}/(k+1)! = o(1/k)$. These results appear in the paper “Asymptotic nature of higher Mahler measure” in Acta Arithmetica [2].

In my next project, generalizing the previous result by an entirely different technique, I established a limit formula $|m_k(P)|/k! \to (1/\pi) \sum_{z_j \in S^1} 1/|P'(z_j)|$ as $k \to \infty$ where the sum on the right hand side is taken over all distinct zeros of $P(z)$ on $S^1$. This result appears in a joint paper with Chris Monico, “Limiting value of higher Mahler measure” in Journal of Number Theory [3].

Both of these results are novel and original. Lehmer’s celebrated conjecture [7] and the appearance of higher Mahler measures in $L$-functions [1, 9] are the main sources of motivation for studying various properties of $m_k(P)$.

After completing these two projects, my goal became to develop a way to calculate higher Mahler measure of any given polynomial. Any method for calculating higher Mahler measure for polynomials other than $P(z) = z + r$ are almost unknown. Use of computer for the same is found to be not useful either.

In this direction, jointly with Ram Murty, I am working to establish a closed form identity for $m_k(P_n)$ where $P_n(z) = z^n - 1 + z^{n-2} + \cdots + z + 1$ for any $n \in \mathbb{N}$. To avoid the direct calculation, we use the generating
function \( Z(s, P_n) := \sum_{k=0}^{\infty} \{m_k(P_n)/k!\} s^k \) of the sequence \( \{m_k(P_n)\}_{k \geq 0} \). Recently, we were able to prove that:

\[
\text{(T1) for all } s \in \mathbb{N}, Z(2s, P_n) \in \mathbb{N}
\]

\[
\text{(T2) } Z(m|s+1|, P_n)/Z(ms, P_n) \to n^m \text{ as } s \to \infty,
\]

and conjectured that:

\[
\text{(C1) } Z(s, n) = f(n, s) \frac{n^s}{\pi} B \left( \frac{s + 1}{2}, \frac{1}{2} \right) \text{ where } B \text{ is the usual Beta function and the “still unknown”}
\]

function \( f(n, s) \) mildly depends on \( n \) and \( s \).

Since the only irreducible factors of the polynomial \( P_n \) are the cyclotomic polynomials, I believe that this will lead me towards a closed form formula of \( m_k(\phi_n) \) where \( \phi_n \) is the \( n \)-th cyclotomic polynomial.

These investigations led me to believe that combining the closed form formula of \( m_k(\phi_n) \) with the celebrated result of Erdős and Turán [6] regarding distribution of roots of polynomials, it is possible to design polynomials whose \( k \)-higher Mahler measure can produce special values of “certain family” of \( L \)-functions.

Since classical Mahler measure has an algebraic interpretation, I would also like to investigate whether it is true for higher Mahler measure as well.

1. Background

In 1933, Lehmer wrote a paper [11] “Factorization of certain cyclotomic functions” in Annals of Math and one of the byproduct of this paper was a method to manufacture large prime numbers. For any \( n \in \mathbb{N} \) and a monic polynomial \( P(z) = \prod_{k=1}^{d} (z - r_k) \in \mathbb{Z}[z] \), he defined a function \( \Delta_n(P) = \prod_{j=1}^{d} (r_j^n - 1) \). By a Galois theoretic argument it can be shown that \( \Delta_n(P) \in \mathbb{Z} \). He was able to produce some huge prime numbers as values of \( \Delta_n(P) \) by suitably choosing \( n \) and \( P \). He noticed that \( \Delta_n(P) \) is more likely to produce primes if its growth rate \( |\Delta_{n+1}(P)/\Delta_n(P)| \) is small. In the same paper he proved that \( \lim_{n \to \infty} |\Delta_{n+1}(P)/\Delta_n(P)| = \prod_{j=1}^{d} \max(1, |r_j|) = \exp(m(P)) \). About thirty years later, Mahler [12] generalized it to multivariate polynomials while searching for tools for transcendence theory. Since small values of the above mentioned growth rate or Mahler measure produce large primes, Lehmer [11] raised the question: Given \( \varepsilon > 0 \), can we find a polynomial \( P(z) \in \mathbb{Z}[z] \) such that \( 0 < m(P) < \varepsilon \)? This is an open question and the smallest known positive measure is that of a polynomial found by Lehmer \( m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 0.1623576120 \ldots \).

It is widely believed that this example represents the true non-trivial minimal value of logarithmic Mahler measure among all integral polynomials and this is known as Lehmer’s Conjecture. Recall that a polynomial \( P(z) \in \mathbb{C}[z] \) is reciprocal if and only if \( P(z) = \pm z^{\deg P} P(z^{-1}) \). Lehmer’s question was answered negatively for non-reciprocal polynomials by Breusch [4] and Smyth [13]. But the above polynomial is a reciprocal one; keeping the conjecture as an open one.

Though the classical Mahler measure was studied extensively, the higher Mahler measure was introduced and studied very recently by Kurokawa, Lalín and Ochiai [9] and Akatsuka [1]. In [10], Lalín and Sinha answered the analogous Lehmer’s Question [7] for higher Mahler measure of order \( k \geq 2 \) by investigating
lower bounds and limit points for higher Mahler measures. They showed the following two results: (a) if \( P(z) \in \mathbb{Z}[z] \), then for \( h \geq 1 \), \( m_{2h}(P) \geq (\pi/12)^h \) if \( P \) is reciprocal and \( m_{2h}(P) \geq (\pi/48)^h \) if \( P \) is non-reciprocal, (b) for \( P_n(z) = z^n + z^{n-1} + \cdots + z + 1 \) and \( h \geq 1 \), \( \lim_{n \to \infty} m_{2h+1}(P_n) = 0 \). Thus, the analogous Lehmer’s Question was answered negatively for higher Mahler measure of even order, and positively for that of odd order for reciprocal polynomials; keeping Lehmer’s question for higher Mahler measure of odd order for non-reciprocal polynomials as (still) unanswered.

It was early noticed (by the works of Smyth [14] and also in [1, 9]) that higher Mahler measure has deep connections with polylogarithms and special values of \( L \)-functions and Riemann zeta functions. For example, \( m_2(z - 1) = \zeta(2)/2 \), \( m_3(z - 1) = -3\zeta(3)/2 \).

2. Completed Research

It is difficult to evaluate higher Mahler measure of order \( k \geq 2 \) for polynomials except for a few specific examples shown in [1] and [9], but it is relatively easy to find their limiting values.

My initial focus was on the monic, linear polynomials \( P(z) = z + r \) where \( r \in S^1 \) and the next two theorems are the main results of my first paper [2].

**Theorem 2.1.** Let \( P(z) = z + r \) where \( r \in S^1 \). Then

\[
\lim_{k \to \infty} \frac{|m_k(P)|}{k!} = \frac{1}{\pi}.
\]

Here I used the Akatsuka’s zeta Mahler measure \( Z(s, P) := \sum_{k=0}^{\infty} \{m_k(P)/k!\} s^k \) of [1] as the generating function of the terms \( m_k(P) \). The proof is based on the derived fact that, \( m_k(P)/k! + m_{k+1}(P)/(k+1)! = O(1/k) \) for \( P \) as defined in the theorem, which in turn allows us to apply the following Littlewood’s version of Tauber’s Theorem [5].

**Theorem 2.2** (Littlewood). Suppose the sequence \( \{k \cdot c_k\}_{k \geq 1} \) is bounded, so that the power series \( g(s) = \sum_{k=0}^{\infty} c_k s^k \) converges in \((-1, 1)\). If \( g(s) \to A \) as \( s \to 1^- \), then the infinite series \( \sum_{k=0}^{\infty} c_k \) converges to \( A \).

on the sequence \( \{m_k(P)/k! + m_{k+1}(P)/(k+1)!\}_{k \geq 1} \) in order to calculate the limit.

The existence of the limit in the above Theorem 2.1 provides us information about the rate of convergence of the sequence \( \{m_k(P)/k!\}_{k \geq 1} \) as summarized in the following theorem.

**Theorem 2.3.** Let \( P(z) = z + r \) where \( r \in S^1 \). Then

\[
\frac{m_k(P)}{k!} + \frac{m_{k+1}(P)}{(k+1)!} = o \left( \frac{1}{k} \right).
\]

Since closed forms of the generating functions of \( m_k(P) \) for polynomials other than of type \( P(z) = z + r \) are still unknown, I could not generalize the limit formula of Theorem 2.1 to any arbitrary polynomial in similar fashion. But using a different technique, I was able to generalize it by computing the same limit for an arbitrary Laurent polynomial \( P(z) \in \mathbb{C}[z, z^{-1}] \) in my next paper [3] with Chris Monico.
Theorem 2.4. Let \( P(z) \in \mathbb{C}[z, z^{-1}] \) be a Laurent polynomial, possibly with repeated roots. Let \( z_1, \ldots, z_n \) be the distinct roots of \( P \). Then
\[
\lim_{k \to \infty} \frac{|m_k(P)|}{k!} = \frac{1}{\pi} \sum_{z_j \in S^1} \frac{1}{|P'(z_j)|}
\]
where \( S^1 \) is the complex unit circle \(|z| = 1\), and the right-hand side is taken as \( \infty \) if \( P'(z_j) = 0 \) for some \( z_j \in S^1 \), i.e., if \( P \) has a repeated root on \( S^1 \).

The proof is based on the simple observation that the integral \( \int_A \log|P(e^{2\pi it})| \, dt \) over any measurable set \( A \subseteq [0, 1] \), where \( A \) is bounded away from the zeros of \( P(e^{2\pi it}) \), grows much slower than \( k! \). The main contributions come from the integrals over the \( \varepsilon \)-neighborhood \((\varepsilon > 0)\) of these zeros on \( S^1 \) and these can be evaluated by linear approximation.

Current Research

There is no simple way to calculate the higher Mahler measure of any polynomial other than \( P(z) = z + r \). It seems very difficult even using a computer. In [10], Lalín and Sinha used the polynomial \( P_n(z) := z^{n-1} + z^{n-2} + \cdots + z + 1 \), but their calculation does not give any easy way for evaluating \( m_k(P_n) \), the \( k \)-higher Mahler measure of \( P_n \).

So, in a joint project with M. Ram Murty, I am working to find a closed form formula for \( m_k(P_n) \). Use of its generating function as \( Z(s, P_n) := \sum_{k=0}^\infty \frac{m_k(P_n)}{k!} s^k = \int_0^1 |\sin(n\pi t)/\sin(\pi t)|^s \, dt \) seems a way to avoid any direct calculation of \( m_k(P_n) \). The following two results, that we found, are rather surprising and encouraging towards our goal of finding a formula for \( m_k(P_n) \).

Theorem 2.5. Define \( P_n(z) := z^{n-1} + z^{n-2} + \cdots + z + 1 \) and \( Z(s, P_n) := \sum_{k=0}^\infty \frac{m_k(P_n)}{k!} s^k \). Let \( s \in \mathbb{N} \) and \( m \in \mathbb{N} \). Then \( Z(2s, P_n) \in \mathbb{N} \).

Theorem 2.6. Let \( P_n \) and \( Z(s, P_n) \) be as described above. Then
\[
\lim_{s \to \infty} \frac{Z(m(s+1), P_n)}{Z(ms, P_n)} = n^m.
\]

We further conjectured that:

Conjecture 2.7. Let \( P_n \) and \( Z(s, P_n) \) be as described above. Then \( Z(s, P_n) = f(s, n) \frac{n^s}{\pi} B \left( \frac{s + 1}{2}, \frac{1}{2} \right) \)
where \( B \) is the usual Beta function and we are still trying to find what the function \( f(n, s) \) is. But we do know that \( f(n, s) \) depends on \( n \) and \( s \) very mildly and \( f(n, s) \to \sqrt{3/(n^2 - 1)} \) as \( s \to \infty \).

As I mentioned earlier, it is quite natural to be hopeful about calculating the higher Mahler measure of cyclotomic polynomials \( (m_k(\phi_n)) \) from a closed form formula of \( m_k(P_n) \), since cyclotomic polynomials \( (\phi_n) \) are the only irreducible factors of \( P_n \).

In [6], Erdős and Turán provided the following beautiful theorem:
Theorem 2.8 (Erdős, Turán). If the roots of the polynomial \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \) are denoted by \( z_\nu = r_\nu e^{i\phi_\nu} \), \( \nu = 1, 2, \ldots, n \) then for every \( 0 \leq \alpha < \beta \leq 2\pi \) we have

\[
\left| \sum_{\alpha \leq \phi_\nu \leq \beta} 1 - \frac{\beta - \alpha}{2\pi} \right| < 16 \sqrt{\frac{n \log |a_0| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}}. \]

I am quite hopeful that this result can be combined with a closed form formula of \( m_k(\phi_n) \) to design polynomials that can produce special values of a family of \( L \)-functions.

References