Exercise Set 6.7

In Exercises 1–6, let C denote the conic section given by the equation. Transform the equation to $x'y'$ coordinates so that C is in standard position with no $x'y'$ term.

1. $27x^2 - 18xy + 3y^2 + x + 3y = 0$
2. $2x^2 - 8xy + 8y^2 + 2x + y = 0$
3. $12x^2 + 8xy + 12y^2 - 8 = 0$
4. $11x^2 - 6xy + 19y^2 + 2x + 4y - 12 = 0$
5. $-x^2 - 6xy - y^2 + 8 = 0$
6. $xy = 1$
7. Let C denote the conic section in standard position given by the equation $4x^2 + 16y^2 = 16$.
   a. Write the quadratic equation in matrix form.
   b. Find the quadratic equation that describes the conic C rotated by $45°$.
8. Let C denote the conic section in standard position given by the equation $x^2 - y^2 = 1$.
   a. Write the quadratic equation in matrix form.
   b. Find the quadratic equation that describes the conic C rotated by $-30°$.
9. Let C denote the conic section in standard position given by the equation $16x^2 + 4y^2 = 16$.
   a. Find the quadratic equation for the conic section obtained by rotating C by $60°$.
   b. Find the quadratic equation that describes the conic found in part (a) after a translation 3 units to the right and 2 units upward.
10. Let C denote the conic section in standard position given by the equation $x^2 - y = 0$.
   a. Find the quadratic equation for the conic section obtained by rotating C by $30°$.
   b. Find the quadratic equation that describes the conic found in part (a) after a translation 2 units to the right and 1 unit downward.

6.8 ▶ Application: Singular Value Decomposition

In earlier sections we have examined various ways to write a given matrix as a product of other matrices with special properties. For example, with the LU factorization of Sec. 1.7, we saw that an $m \times n$ matrix A could be written as $A = LU$ with L being an invertible lower triangular matrix and U an upper triangular matrix. Also in Sec. 1.7, we showed that if A is invertible, then it could be written as the product of elementary matrices. In Sec. 5.2 it was shown that an $n \times n$ matrix A with n linearly independent eigenvectors can be written as

$$A = PDP^{-1}$$

where D is a diagonal matrix of eigenvalues of A. As a special case, if A is symmetric, then A has the factorization

$$A = QDQ^t$$

where Q is an orthogonal matrix.

In this section we consider a generalization of this last result for $m \times n$ matrices. Specifically, we introduce the *singular value decomposition*, abbreviated as SVD, which enables us to write any $m \times n$ matrix as

$$A = U\Sigma V^t$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and Σ is an $m \times n$ matrix with numbers, called *singular values*, on its diagonal.

Singular Values of an $m \times n$ Matrix

To define the singular values of an $m \times n$ matrix A, we consider the matrix A^tA. Observe that since A is an $m \times n$ matrix, A^t is an $n \times m$ matrix, so the product A^tA is a square $n \times n$ matrix. This new matrix is symmetric since $(A^tA)^t = A^tA^{tt} = A^tA$. Hence, by Theorem 14 of Sec. 6.6, there is an orthogonal matrix P such that

$$P^t(A^tA)P = D$$

where D is a diagonal matrix of the eigenvalues of A^tA given by

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Since by Exercise 39 of Sec. 6.3 the matrix A^tA is positive semidefinite, we also have, by Exercise 41 of Sec. 6.3, that $\lambda_i \geq 0$ for $1 \leq i \leq n$. This permits us to make the following definition.

DEFINITION 1 **Singular Values** Let A be an $m \times n$ matrix. The **singular values** of A, denoted by σ_i for $1 \leq i \leq n$, are the positive square roots of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A^tA. That is,

$$\sigma_i = \sqrt{\lambda_i} \qquad \text{for} \quad 1 \leq i \leq n$$

It is customary to write the singular values of A in decreasing order

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$

As mentioned in Sec. 5.2, this can be accomplished by permuting the columns of the diagonalizing matrix P.

EXAMPLE 1 Let A be the matrix given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find the singular values of A.

Solution The singular values of A are found by first computing the eigenvalues of the square matrix

$$A^tA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation, in this case, is given by

$$\det(A^t A - \lambda I) = (\lambda - 3)(\lambda - 1) = 0$$

The eigenvalues of $A^t A$ are then $\lambda_1 = 3$ and $\lambda_2 = 1$, so that the singular values are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

We have already seen that orthogonal bases are desirable and the Gram-Schmidt process can be used to construct an orthogonal basis from any basis. If A is an $m \times n$ matrix and $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are the eigenvectors of $A^t A$, then we will see that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for $\mathbf{col}(A)$. We begin with the connection between the singular values of A and the vectors $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$.

THEOREM 16 Let A be an $m \times n$ matrix and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^t A$, with corresponding eigenvectors $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

1. $\| A\mathbf{v}_i \| = \sigma_i$ for each $i = 1, 2, \ldots, n$.
2. $A\mathbf{v}_i$ is orthogonal to $A\mathbf{v}_j$ for $i \neq j$.

Proof For the first statement recall from Sec. 6.6 that the length of a vector $\mathbf{v}$ in Euclidean space can be given by the matrix product $\| \mathbf{v} \| = \sqrt{\mathbf{v}^t \mathbf{v}}$. Therefore,

$$\| A\mathbf{v}_i \|^2 = (A\mathbf{v}_i)^t (A\mathbf{v}_i) = \mathbf{v}_i^t (A^t A)\mathbf{v}_i = \mathbf{v}_i^t \lambda_i \mathbf{v}_i = \lambda_i \| \mathbf{v}_i \| = \lambda_i$$

The last equality is due to the fact that $\mathbf{v}_i$ is a unit vector. Part 1 is established by noting that $\sigma_i = \sqrt{\lambda_i} = \| A\mathbf{v}_i \|$. For part 2 of the theorem, we know that (as in Sec. 6.6) the dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ in Euclidean space can be given by the matrix product $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^t \mathbf{v}$. Thus, since B is an orthonormal basis of $\mathbb{R}^n$, if $i \neq j$, then

$$(A\mathbf{v}_i) \cdot (A\mathbf{v}_j) = (A\mathbf{v}_i)^t (A\mathbf{v}_j) = \mathbf{v}_i^t (A^t A)\mathbf{v}_j = \mathbf{v}_i^t \lambda_j \mathbf{v}_i = \lambda_j \mathbf{v}_i^t \mathbf{v}_j = 0$$

In Theorem 16, the set of vectors $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is shown to be orthogonal. In Theorem 17 we establish that the eigenvectors of $A^t A$, after multiplication by A, are an orthogonal basis for $\mathbf{col}(A)$.

THEOREM 17 Let A be an $m \times n$ matrix and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^t A$. Suppose that the corresponding eigenvalues satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots \lambda_n = 0$, that is, $A^t A$ has r nonzero eigenvalues. Then $B' = \{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for the column space of A and $\mathbf{rank}(A) = r$.

Proof First observe that since $\sigma_i = \sqrt{\lambda_i}$ are all nonzero for $1 \le i \le r$; then by Theorem 16, part 1, we have $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r$ are all nonzero vectors in $\mathbf{col}(A)$. By part 2 of Theorem 16, we have $B' = \{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r\}$ is an orthogonal set of vectors in $\mathbb{R}^m$. Hence, by Theorem 5 of Sec. 6.2, B' is linearly independent. Now to show that these vectors span the column space of A, let $\mathbf{w}$ be a vector in $\mathbf{col}(A)$. Thus, there exists a vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{w}$. Since $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, there are scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Multiplying both sides of the last equation by A, we obtain

$$A\mathbf{v} = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_n A\mathbf{v}_n$$

Now, using the fact that $A\mathbf{v}_{r+1} = A\mathbf{v}_{r+2} = \cdots = A\mathbf{v}_n = \mathbf{0}$, then

$$A\mathbf{v} = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_r A\mathbf{v}_r$$

so that $\mathbf{w} = A\mathbf{v}$ is in $\mathbf{span}\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r\}$. Consequently, $B' = \{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for the column space of A, and the rank of A is equal to the number of its nonzero singular values.

EXAMPLE 2 Let A be the matrix given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find the image of the unit circle under the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\mathbf{v}) = A\mathbf{v}$.

Solution From Example 1, the eigenvalues of A^tA are $\lambda_1 = 3$ and $\lambda_2 = 1$, with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

respectively. The singular values of A are then $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$. Let $C(t)$ be the unit circle given by $\cos(t)\mathbf{v}_1 + \sin(t)\mathbf{v}_2$ for $0 \le t \le 2\pi$. The image of $C(t)$ under T is given by

$$T(C(t)) = \cos(t)A\mathbf{v}_1 + \sin(t)A\mathbf{v}_2$$

By Theorem 17, $B' = \left\{\frac{1}{\sigma_1}A\mathbf{v}_1, \frac{1}{\sigma_2}A\mathbf{v}_2\right\}$ is a basis for the range of T. Hence, the coordinates of $T(C(t))$ relative to B' are $x' = \sigma_1 \cos t = \sqrt{3}\cos t$ and $y' = \sigma_2 \sin t = \sin t$. Observe that

$$\left(\frac{x'}{\sqrt{3}}\right)^2 + (y')^2 = \frac{(x')^2}{3} + (y')^2 = \cos^2 t + \sin^2 t = 1$$

which is an ellipse with the length of the semimajor axis equal to σ_1 and length of the semiminor axis equal to σ_2, as shown in Fig. 1.

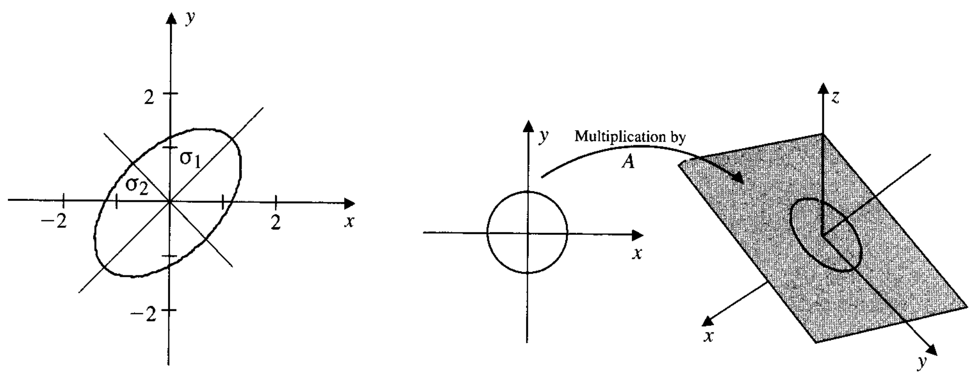

Figure 1

For certain matrices, some of the singular values may be zero. As an illustration, consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. For this matrix, we have $\mathbf{col}(A) = \mathbf{span}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The reduced row echelon form for A is the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, which has only one pivot column. Hence, the rank of A is equal to 1. The eigenvalues of A^tA are $\lambda_1 = 50$ and $\lambda_2 = 0$ with corresponding unit eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

The singular values of A are given by $\sigma_1 = 5\sqrt{2}$ and $\sigma_2 = 0$. Now, multiplying $\mathbf{v}_1$ and $\mathbf{v}_2$ by A gives

$$A\mathbf{v}_1 = \begin{bmatrix} \sqrt{5} \\ 3\sqrt{5} \end{bmatrix} \quad \text{and} \quad A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Observe that $A\mathbf{v}_1$ spans the one dimensional column space of A. In this case, the linear transformation $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps the unit circle to the line segment

$$\left\{ t\begin{bmatrix} \sqrt{5} \\ 3\sqrt{5} \end{bmatrix} \middle| -1 \le t \le 1 \right\}$$

as shown in Fig. 2.

Singular Value Decomposition (SVD)

We now turn our attention to the problem of finding a singular value decomposition of an $m \times n$ matrix A.

THEOREM 18 **SVD** Let A be an $m \times n$ matrix of rank r, with r nonzero singular values $\sigma_1, \sigma_2, \ldots, \sigma_r$. Then there exists an $m \times n$ matrix Σ, an $m \times m$ orthogonal matrix

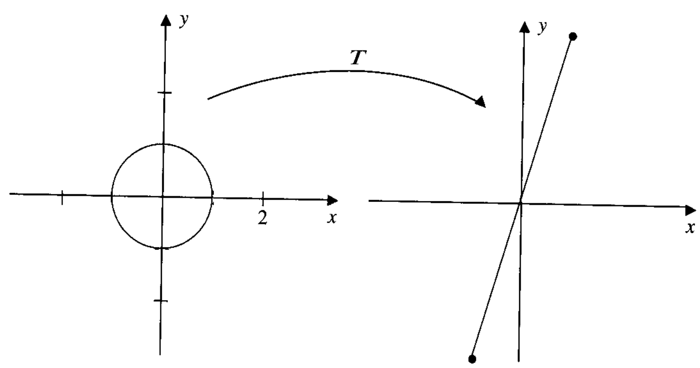

Figure 2

U, and an $n \times n$ orthogonal matrix V such that

$$A = U\Sigma V^t$$

Proof Since A^tA is an $n \times n$ symmetric matrix, by Theorem 14 of Sec. 6.6 there is an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$, consisting of eigenvectors of A^tA. Now by Theorem 17, $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for $\mathbf{col}(A)$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be the orthonormal basis for $\mathbf{col}(A)$, given by

$$\mathbf{u}_i = \frac{1}{\| A\mathbf{v}_i \|} A\mathbf{v}_i = \frac{1}{\sigma_i} A\mathbf{v}_i \qquad \text{for} \qquad i = 1, \ldots, r$$

Next, extend $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ to the orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of $\mathbb{R}^m$. We can now define the orthogonal matrices V and U, using the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, respectively, as column vectors, so that

$$V = [\ \mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\ ] \qquad \text{and} \qquad U = [\ \mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_m\ ]$$

Moreover, since $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$, for $i = 1, \ldots, r$, then

$$AV = \begin{bmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \sigma_1\mathbf{u}_1 & \cdots & \sigma_r\mathbf{u}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

Now let Σ be the $m \times n$ matrix given by

$$\Sigma = \left[\begin{array}{cccc|ccc} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \sigma_r & 0 & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]$$

Then

$$U\Sigma = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{bmatrix} \Sigma$$

$$= \begin{bmatrix} \sigma_1\mathbf{u}_1 & \cdots & \sigma_r\mathbf{u}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

$$= AV$$

Since V is orthogonal, then $V^t = V^{-1}$, and hence, $A = U\Sigma V^t$.

EXAMPLE 3 Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$$

Solution A procedure for finding an SVD of A is included in the proof of Theorem 18. We present the solution as a sequence of steps.

Step 1. *Find the eigenvalues and corresponding orthonormal eigenvectors of A^tA and define the matrix V.*

The eigenvalues of the matrix

$$A^tA = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}$$

in decreasing order are given by $\lambda_1 = 12$ and $\lambda_2 = 0$. The corresponding orthonormal eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Since the column vectors of V are given by the orthonormal eigenvectors of A^tA, the matrix V is given by

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Step 2. *Find the singular values of A and define the matrix Σ.*

The singular values of A are the square roots of the eigenvalues of A^tA, so that

$$\sigma_1 = \sqrt{\lambda_1} = 2\sqrt{3} \quad \text{and} \quad \sigma_2 = \sqrt{\lambda_2} = 0$$

Since Σ has the same dimensions as A, then Σ is 3×2. In this case,

$$\Sigma = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Step 3. *Define the matrix* U.
The matrix A has one nonzero singular value, so by Theorem 17 the rank of A is 1. Therefore, the first column of U is

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

Next we extend the set $\{\mathbf{u}_1\}$ to an orthonormal basis for $\mathbb{R}^3$ by adding to it the vectors

$$\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

so that

$$U = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & -1/\sqrt{2} \\ 1/\sqrt{6} & 0 & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{5} & 0 \end{bmatrix}$$

The singular value decomposition of A is then given by

$$A = U\Sigma V^t = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & -1/\sqrt{2} \\ 1/\sqrt{6} & 0 & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$$

In Example 3, the process of finding a singular value decomposition of A was complicated by the task of extending the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ to an orthogonal basis for $\mathbb{R}^m$. Alternatively, we can use A^tA to find V and AA^t to find U. To see this, note that if $A = U\Sigma V^t$ is an SVD of A, then $A^t = V\Sigma^t U^t$. After multiplying A on the left by its transpose, we obtain

$$A^tA = V\Sigma^t U^t U\Sigma V^t = VD_1V^t$$

where D_1 is an $n \times n$ diagonal matrix with diagonal entries the eigenvalues of A^tA. Hence, V is an orthogonal matrix that diagonalizes A^tA. On the other hand,

$$AA^t = U\Sigma V^t V\Sigma^t U^t = UD_2U^t$$

where D_2 is an $m \times m$ diagonal matrix with diagonal entries the eigenvalues of AA^t and U is an orthogonal matrix that diagonalizes AA^t. Note that the matrices A^tA and AA^t have the same eigenvalues. (See Exercise 22 of Sec. 5.1.) Therefore, the nonzero diagonal entries of D_1 and D_2 are the same. The matrices U and V found using this procedure are not unique. We also note that changing the signs of the column vectors in U and V also produces orthogonal matrices that diagonalize AA^t and A^tA. As a result, finding an SVD of A may require changing the signs of certain columns of U or V.

In Example 4 we use this idea to find an SVD for a matrix.

EXAMPLE 4 Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}$$

Solution First observe that

$$A^tA = \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 10 \end{bmatrix}$$

By inspection we see that $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a unit eigenvector of A^tA with corresponding eigenvalue $\lambda_1 = 18$, and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a unit eigenvector of A^tA with corresponding eigenvalue $\lambda_2 = 2$. Hence,

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The singular values of A are $\sigma_1 = 3\sqrt{2}$ and $\sigma_2 = \sqrt{2}$ so that

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

To find U, we compute

$$AA^t = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 18 \end{bmatrix}$$

Observe that a unit eigenvector corresponding to $\lambda_1 = 18$ is $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and a unit eigenvector corresponding to $\lambda_2 = 2$ is $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus,

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A singular value decomposition of A is then given by

$$A = U\Sigma V^t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}$$