RESEARCH STATEMENT

TRISTRAM BOGART

1. Introduction

The intersection of commutative algebra and algebraic geometry with combinatorics and discrete geometry is a rich and expanding field. Integral polytopes, purely discrete objects, provide enlightening examples to algebraic geometers via the theory of *toric varieties*. More recently, the method of *tropical geometry* has been developed to better understand more general algebraic varieties through polyhedral geometry. Conversely, structural results in the theory of convex polytopes such as the *g-theorem* have been proved using commutative algebra. My thesis work consists of three projects that bridge this gap: one in commutative algebra, one in tropical geometry, and the third in discrete geometry and integer programming. In the second section of this document I summarize each of these projects. In the third and final section I discuss my current research projects and future plans.

Key ingredients in my research include:

- **Gröbner bases.** Gröbner bases of ideals in polynomial rings are a basic tool in computational commutative algebra and algebraic geometry. Gröbner bases were implicit in the work of Hilbert and Macaulay but gained prominence when Buchberger [Buc65] presented an algorithm to compute them that now bears his name. Geometrically, a Gröbner basis defines a deformation from an arbitrary algebraic variety to a union of coordinate subspaces that carries the same arithmetic invariants such as dimension and Hilbert function. More recently, Mora and Robbiano [MR88] defined the *Gröbner fan*, a discrete object that parametrizes all possible Gröbner bases of a given ideal. Conti and Traverso [CT91] and Sturmfels [Stu96] developed the theory of Gröbner bases and Gröbner fans for the important special case of toric ideals and much of my work is in this area.
- **Enumeration and Complexity.** Exact and asymptotic counting methods are central to discrete mathematics. Generating functions allow apparently complex sequences of integers to be represented by much simpler rational expressions and can be used to prove difficult and surprising identities. Recent work of Barvinok and Woods [BW96] shows the effectiveness of generating functions in enumerating integer points inside a polytope of fixed dimension and has been extended [LHH$^+$04] to a range of other problems including Gröbner bases of toric ideals. Computational complexity theory and NP-completeness, based on work of Cook [Coo71] and many others that shows the equivalence of a wide variety of difficult algorithmic problems, provides the framework for the Barvinok-Woods results as well as for some of the questions in my thesis and in my current work on Hom polytopes.
- **Computer experimentation.** Much of my research has involved the use of the computer algebra system Macaulay 2 [GS] and specialized packages such as Gfan [Jen] for Gröbner fans and tropical varieties and Polymake [GJ] for polyhedral computations. These methods

have allowed me to formulate conjectures that I would not otherwise have considered and also to illustrate my work with a range of nontrivial examples.

2. Thesis Work

2.1. **Tropical Varieties.** One part of my thesis involves *tropical geometry* and derives from joint work [BJS$^+$07] with Anders Jensen, David Speyer, Bernd Sturmfels, and Rekha Thomas. To every subvariety V of the algebraic torus $(\mathbb{C}^*)^n$ can be associated a polyhedral complex, its *tropical variety* $\mathcal{T}(V)$ [Ber71, BG84, RGST03]. This object satisfies standard algebro-geometric relations such as Bezout's theorem that apply to V itself. Tropical varieties have been applied to time-dependent systems of polynomial equations arising in celestial mechanics [HM96], to algebraic dynamical systems [EKL06], and to string theory [Ray] as well as to purely algebro-geometric problems.

To define them, let $K := \mathbb{C}\{\{t\}\}$ be the field of *Puiseux series:* power series in t with rational exponents with bounded denominators. This field has a natural valuation:

$$\deg(c_b t^b + \text{higher order terms}) = b.$$

Given a polynomial $f(x_1, \ldots, x_n) = \sum_{\mathbf{a}} c_{\mathbf{a}} x_1^{a_1} \cdots x_n^{a_n}$ over K, its *tropicalization* is the piecewise linear function

$$F(X_1, \ldots, X_n) = \min_{\mathbf{a}}(\deg(c_{\mathbf{a}}) + a_1 X_1 + \ldots + a_n X_n).$$

The *tropical hypersurface* of f, $\mathbb{T}(f)$, is the subset of $\mathbb{R}^n$ on which F is nondifferentiable; that is, the minimum is achieved by at least two terms. If I is an ideal in $K[x_1, \ldots x_n]$, then the tropical variety of I, $\mathbb{T}(I)$, is the intersection of the tropical hypersurfaces $\mathbb{T}(f)$ as f runs over all polynomials in I. A finite set of polynomials $\{f_1, \ldots, f_t\}$ is a *tropical basis* of I if $\langle f_1, \ldots, f_t \rangle = I$ and $\mathbb{T}(I) = \mathbb{T}(f_1) \cap \cdots \cap \mathbb{T}(f_t)$.

We presented an algorithm to compute tropical varieties and illustrated its implementation in the software package `Gfan` [Jen]. To prove correctness of the algorithm, we showed [BJS$^+$07, Theorem 3.1] that any irreducible tropical variety $\mathbb{T}(I)$ is connected in codimension one. A family of examples [BJS$^+$07, Theorem 6.3] demonstrated that the algorithm implemented in `Gfan` is far superior to the naive one of searching the entire Gröbner fan.

We also showed [BJS$^+$07, Theorem 2.9] that every ideal has a tropical basis. However [BJS$^+$07, Theorem 2.10], for every d and n we identified a linear ideal I of Krull dimension d in $\mathbb{C}[x_1, \ldots, x_n]$ such that the size of any tropical basis of I consisting of linear forms is at least $\frac{1}{n-d+1}\binom{n}{d}$, answering a question of Speyer and Sturmfels [SS04, §5]. More recently, Theobald and Hept [HT, Theorem 1.1] showed in contrast that every prime ideal I of codimension $n - d$ and generated by r polynomials has a tropical basis of size at most $n - d + 1 + r$. Counterintuitively, the smallest tropical basis of a linear ideal need not consist of linear forms.

2.2. **Circuit Ideals.** Another part of my thesis is the study of *circuit ideals*, joint work with Anders Jensen and Rekha Thomas [BJT07]. Given a finite set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$, a *circuit* of $\mathcal{A}$ is a nonzero vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ such that $\sum c_i \mathbf{a}_i = 0$ and whose set of nonzero components is minimal among those with this property. The *circuit ideal* of $\mathcal{A}$ is the ideal generated by the binomials $\mathbf{x}^{\mathbf{c}^+} - \mathbf{x}^{\mathbf{c}^-} \in \mathbb{C}[x_1, \ldots, x_n]$ over all circuits $\mathbf{c}$ of $\mathcal{A}$. Its radical is the *toric ideal* of $\mathcal{A}$, generated by the binomials $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ varies over all vectors in $\mathbb{Z}^n$ such that $\sum u_i \mathbf{a}_i = 0$. Toric ideals are the defining ideals of *toric varieties* [Ful93] and have applications to combinatorics, optimization, algebra, and algebraic geometry [Stu96]. However, they are difficult to compute. Since circuits can be computed using linear algebra and the two ideals often coincide, it is worthwhile to understand when the two ideals are equal.

We characterized [BJT07, Theorem 3.21] when equality occurs in terms of the existence of certain polytopes and their integer points. When the two ideals are not equal, we interpreted the primary decomposition of the circuit ideal in terms of the difference between their respective Gröbner bases [BJT07, Theorem 4.4]. We also showed [BJT07, Theorem 5.16] that in dimension two but not in dimension three or more, the Gröbner fan of the circuit ideal always refines that of the toric ideal.

2.3. **Small Chvátal Rank.** The remaining part of my thesis consists of joint work [BT] with Rekha Thomas in the theory of polytopes and *integer programming*, the problem of finding an (optimal) integer solution to a system of linear inequalities; that is, finding an integer point in a polytope P. This problem provides an important application of Gröbner bases and Gröbner fans of toric ideals [Stu96, §5]. Given a system of inequalities $A\mathbf{x} \leq \mathbf{b}$ defining P, the *Chvátal procedure* is an iterative method to obtain the *integer hull* of P; that is, the convex hull of its integer points. The number of iterations required is known as the *Chvátal rank* of the system and is known to be finite [Sch86, §23]. However, Chvátal rank is not just a function of the dimension of the polytope P and can be exponentially large, even in dimension two.

Using a variant of the Chvátal procedure that depends only the matrix A and not any particular right-hand-side $\mathbf{b}$, we defined a new parameter, the *small Chvátal rank* (SCR) of the system $A\mathbf{x} \leq \mathbf{b}$ and of A. This number is at most the Chvátal rank but in certain cases it is much smaller. In dimension two the SCR is always at most one, while Chvátal rank can be arbitrarily large. The SCR of the *coclique polytope* of a complete graph K_n is one or two depending on the parity of n, while Chvátal rank is known to be $\mathcal{O}(\log n)$.

However, the SCR of a 3×3 matrix, like its Chvátal rank, can be exponentially large in the input size. We also treated the important case of polytopes contained in the unit cube C_n. Eisenbrand and Schulz [ES03] have shown that the Chvátal ranks of polytopes in the unit cube, an important class for combinatorial optimization, are at most $n^2(1+\log n)$ and can be as large as $(1+\epsilon)n$. Using different techniques, we showed that the SCR of such polytopes can be as large as $n/2$.

3. Current and Future Research

3.1. **Tropical rational curves in hypersurfaces.** In ongoing work with fellow Queen's post-doc Ethan Cotterill, I'm applying tropical geometry to the study of rational curves in generic hypersurfaces in the projective space $\mathbb{P}^n$. In general, it is known that hypersurfaces of low degree contain many rational curves and that those of high degree contain none, but a key boundary case is that of *Calabi-Yau* hypersurfaces: those of degree $m = n+1$. Calabi-Yau varieties are the setting for the theory of *mirror symmetry* [CK99] used by string theorists. Herbert Clemens [Cle68] showed that for $m \geq 2n-1$, a generic hypersurface of degree M contains no rational curves; this result was later extended to the case $n = 4$, $m = 2n-2$ by Claire Voisin. Still open is Clemens' conjecture that a generic Calabi-Yau hypersurface in $\mathbb{P}^4$ (i.e. $m = 5$) contains only finitely many rational curves in each degree.

Inspired by Mikhalkin's development of tropical enumerative geometry [Mik05], its use in proving the Kontsevich formula [GM08], and by Vigeland's work [Vig] on the enumeration of tropical lines in tropical surfaces in $\mathbb{R}^3$, we set out to approach this area tropically. In particular, our first aim is to prove the tropical analogue of Clemens' result in dimension three.

Tropical hypersurfaces of degree m occur in many different combinatorial types. We follow Vigeland's choice of hypersurfaces, derived from a highly symmetric unimodular triangulation of the standard tetrahedron of edge length m in $\mathbb{R}^3$. If a generic tropical hypersurface of a given degree m in $\mathbb{R}^n$ contains no tropical rational curves, then the same holds for ordinary curves. Thus the tropical analogue of Clemens' result would imply the original result. Furthermore, a tropical proof would improve on the original complex geometric proof by being characteristic-free.

Tropical curves are weighted graphs embedded in $\mathbb{R}^n$, with some edges allowed to be unbounded rays, that satisfy the *zero-tension property*: the weighted sum of all primitive integer edge directions at each vertex is zero. Speyer [Spe] has shown that if $\mathcal{C}$ is a rational curve over the Puiseux series field K, then $\mathcal{T}(C)$ is a tree (i.e., its genus as a topological space is zero) and conversely that every zero-tension tree arises as the tropicalization of such a curve. So our goal amounts to showing that certain two-dimensional polyhedral complexes do not contain any zero-tension trees. If $\mathcal{C}$ is chosen generically over K, then $\mathcal{T}(C)$ is a *caterpillar graph*: a path with a ray attached to each of its internal nodes and two rays at each end. We have shown that Vigeland's surfaces contain no balanced caterpillar graph for m sufficiently large; that is, they contain no *generic* tropical rational curves.

Our main goals are to extend our methods to all zero-tension trees in Vigeland's surfaces, to apply them to tropical hypersurfaces in higher-dimensional space, based on an analogue to Vigeland's triangulation for higher-dimensional simplices, and to prove finiteness results that might lead toward Clemens' conjecture. For a tropical hypersurface to contain only finitely many rational curves of a given degree, every tree with the appropriate number of rays would need to be *rigid*. This would be the case if many vertices of each tree coincide with vertices of the tropical hypersurface itself.

3.2. **Hom polytopes.** In the summer of 2008, Joseph Gubeladze invited me to visit him at San Francisco State University, and we began a new research project on *Hom polytopes.* Ziegler [Zie95, §9] observes that given full-dimensional polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$, the set $\mathrm{Hom}(P, Q)$ of affine maps ϕ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $\phi(P) \subseteq Q$ is itself a convex polytope and that it has an explicit facet description: for each vertex v of P and each facet H of Q, the constraint that $\phi(v)$ lies on the correct side of H provides a facet inequality to $\mathrm{Hom}(P, Q)$. He also suggests that Hom polytopes, combined with the *fiber polytopes* of Billera and Sturmfels [BS92], could inaugurate a categorical theory of polytopes.

Since little is known about Hom polytopes, we begin our study with the case where P is a regular p-gon and Q a regular q-gon. Our goal is to understand the f-*vector* (the number of faces of each dimension) of $H := Hom(P, Q)$ in this case. A map ϕ in H consists of a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ followed by a translation. In the Euclidean topology, an open subset of the set of such affine maps lies in $\mathrm{Hom}(P, Q)$, so $\mathrm{Hom}(P, Q)$ is a six-polytope. The vertices ϕ of H are naturally stratified into three sets V_0, V_1, V_2 by the dimension of the image of $\phi(P)$. The vertices in V_0 are simply the q

maps where all of $\mathbb{R}^n$ is sent to one of the vertices of q. We have shown that

$$|V_1| = \begin{cases} p\binom{q}{2} \text{ for } p \text{ odd} \\ \frac{p}{2}\binom{q}{2} \text{ for } p \text{ even} \end{cases}$$

by identifying all of the maps in this set. From examples computed with Polymake and from our intuition about "typical" polygons, we conjecture that if p and q are coprime, than every vertex of H in V_2 is simple: i.e, it lies on exactly six facets, the minimum number possible. Since we can also show that (asymptotically) most of the vertices of H are in V_2, this would imply that the f-vector of H is close to that of a *simple polytope*; one whose vertices are all simple. The range of f-vectors of simple polytopes is described by the *g-theorem* of Stanley [Sta80] and Billera-Lee [BL81]. However, our conjecture cannot extend to all p and q, as for example the identity map from a regular p-gon to itself is a vertex in V_2 that lies on $2p$ facets of H.

3.3. **Tropical polytopes and cellular resolutions.** In the future, I am interested in studying the connections between tropical geometry and the theory of free resolutions of monomial ideals. A *free resolution* [Eis05] of a homogeneous ideal I in the polynomial ring $S := \mathbb{C}[x_1, \ldots, x_n]$ is an exact sequence

$$0 \to F_k \xrightarrow{\phi_k} F_{k-1} \xrightarrow{\phi_{k-1}} \ldots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \simeq S$$

of free S-modules such that $\mathrm{coker}(\phi_1) \simeq S/I$. For each i and j, the *Betti number* $\beta_{i,j}$ of I is the dimension of the degree-j component of the module F_i, and the *Hilbert function* of I can be directly computed from the Betti numbers. Monomial ideals have the advantage of being combinatorially specified: they are generated by a finite list of monomials $\mathbf{x}^{\mathbf{m}_1}, \ldots, \mathbf{x}^{\mathbf{m}_s}$, each identified with its exponent vector in $\mathbb{N}^n$. Furthermore, any ideal I in S can be deformed to a monomial ideal J by the computation of a *Gröbner basis*. The ideals I and J need not have isomorphic resolutions, but do have the same Hilbert function. Bayer and Sturmfels [BS98] introduced the notion of a *cellular resolution:* a cell complex X whose cells of dimension k are indexed by the (multi)degrees of the generators of the kth module F_k in a free resolution of J, and whose boundary maps are the maps in the resolution. Minimal cellular resolutions have been constructed for various classes of monomial ideals [MS04], yet there exist monomial ideals with no minimal cellular resolution at all [Vel08].

Develin and Yu [DY07] recently observed that the surfaces supporting cellular resolutions of generic monomial ideals can be interpreted as *tropical polytopes.* Since resolutions of monomial ideals can be efficiently computed with software such as Macaulay 2 [GS] or CoCoa [coc], their observation provides a new computational tool for tropical geometry. A tropical polytope [Jos05] can be defined either as the "convex hull" of finitely many points in $\mathbb{R}^n$ under the tropical operations, or as the image of a polytope in K., where K is again the field of Puiseux series. The latter definition mimics one of the usual definitions of tropical variety. However, only certain tropical polytopes arise from these resolutions: those whose vertices are in general position with respect to the tropical operations. As Develin and Yu point out, the theory of tropical polytopes is not well-developed beyond this case; it is not even clear how the faces should be defined.

The goal of this project would be to exploit the connection between cellular resolutions and tropical polytopes to enrich both areas. In particular, Ezra Miller suggests the problem of extending the resolution technique for *rigid* monomial ideals [Mil02] from three to higher dimensions. The surfaces supporting these resolutions are locally tropically convex. Using examples derived from monomial ideals, it may be possible to elucidate the definition of tropical polytopes: the faces of the polytopes should be cells supporting components of the modules in the resolution. Furthermore, f-vectors of these polytopes could be constrained by known results on the allowed Betti numbers of the resolutions of various types of ideals. For example, Gruson, Lazarsfeld and Peskine proved a sharp bound [Eis05, Theorem 5.1] on the *regularity* of the ideal of an algebraic curve, and the Eisenbud-Goto conjecture would generalize this bound to varieties of higher dimension. What are the implications for the combinatorics of tropical polytopes?

References

[Ber71] George Bergman. The logarithmic limit set of an algebraic variety. *Trans. Amer. Math. Soc.*, 157:459–469, 1971.

[BG84] Robert Bieri and J.R.J. Groves. The geometry of the set of characters induced by valuations. *J. Reine Angew. Mathematik*, 347:168–195, 1984.

[BJS+07] Tristram Bogart, Anders N. Jensen, David Speyer, Bernd Sturmfels, and Rekha R. Thomas. Computing tropical varieties. *Journal of Symbolic Computation*, 42(1-2):54–73, 2007.

[BJT07] Tristram Bogart, Anders N. Jensen, and Rekha R. Thomas. The ideal of circuits of a vector configuration. *Journal of Algebra*, 309(2):518–542, 2007.

[BL81] L. J. Billera and C. W. Lee. A proof of the sufficiency of mcmullen's conditions for f-vectors of simplicial polytopes. *Journal of Combinatorial Theory* ***A***, 31:237–255, 1981.

[BS92] L. J. Billera and Bernd Sturmfels. Fiber polytopes. *Annals of Mathematics*, 135:527–549, 1992.

[BS98] Dave Bayer and Bernd Sturmfels. Cellular resolutions of monomial modules. *Journal für die Reine und Angewandte Mathematik*, 502:123–140, 1998.

[BT] T. Bogart and R.R. Thomas. Small Chvatal rank. submitted to *Mathematical Programming*; arXiv:0705.1027.

[Buc65] B. Buchberger. *On Finding a Vector Space Basis of the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal*. PhD thesis, University of Innsbruck, Austria, 1965. (German).

[BW96] A. Björner and M. Wachs. Shellable nonpure complexes and posets i. *Trans. Amer. Math. Soc.*, 348:1299–1327, 1996.

[CK99] David Cox and Sheldon Katz. *Mirror Symmetry and Algebraic Geometry*, volume 68. Mathematical Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, 1999.

[Cle68] Herbert Clemens. Curves on generic hypersurfaces. *Ann. Sci. École Norm. Sup. (4)*, 19(4):629–636, 1968.

[coc] CoCoA 4.1. Available from ftp://cocoa.dima.unige.it/cocoa.

[Coo71] S.A. Cook. The complexity of theorem-proving procedures. In *Proceedings, Third Annual ACM Symposium on the Theory of Computing*, pages 151–158, New York, 1971. ACM.

[CT91] P. Conti and C. Traverso. Buchberger algorithm and integer programming. In H. F. Mattson, T. Mora, and T. R. N. Rao, editors, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 130–139. Lecture Notes in Computer Science 539, Springer-Verlag, 1991.

[DY07] Mike Develin and Josephine Yu. Tropical polytopes and cellular resolutions. *Experimental Mathematics*, 16:277–291, 2007.

[Eis05] David Eisenbud. *The Geometry of Syzygies*. Springer Graduate Texts in Mathematics, 2005.

[EKL06] Manfried Einsiedler, Mikhail Kapranov, and Douglas Lind. Non-Archimedean amoebas and tropical varieties. *Journal für die Reine und Angewandte Mathematik*, 601:139–157, 2006.

[ES03] F. Eisenbrand and A.S. Schulz. Bounds on the chvátal rank of polytopes in the 0/1 cube. *Combinatorica*, 23(2):245–261, 2003.

[Ful93] W. Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.

[GJ] E. Gawrilow and M. Joswig. Polymake. available from http://www.math.tu-berlin.de/polymake/.

[GM08] Andreas Gathmann and Hannah Markwig. Kontsevich's formula and the wdvv equations in tropical geometry. *Advances in Mathematics*, 217:537–560, 2008.

[GS] Daniel R. Grayson and Mike Stillman. Macaulay 2, a software system for research in algebraic geometry. Available from http://www.math.uiuc.edu/Macaulay2.

[HM96] Marshall Hampton and Rick Moeckel. Finiteness of relative equilibria of the four- body problem. *Inv. Math.*, 163:289–312, 20096.

[HT] K. Hept and T. Theobald. Tropical bases by regular projections. arXiv:0708.1727v1.

[Jen] Anders N. Jensen. Gfan, a software system for gröbner fans. available from http://home.imf.au.dk/ajensen/software/gfan/gfan.html.

[Jos05] Michael Joswig. Tropical convexity. In *Combinatorial and Computational Geometry*, volume 52, pages 409–431, Cambridge, 2005. Math. Sci. Res. Inst. Publ., Cambridge Univ. Press.

[LHH$^+$04] J.A. De Loera, D. Haws, R. Hemmecke, R. Huggins, and Bernd Sturmfels. Short rational functions for toric algebra and applications. *Journal of Symbolic Computation*, 38(4):1273–1302, 2004.

[Mik05] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *Journal of the American Mathematical Society*, 18:313–377, 2005.

[Mil02] Ezra Miller. Planar graphs as minimal resolutions of trivariate monomial ideals. *Documenta Mathematica*, 7:43–90, 2002.

[MR88] T. Mora and Lorenzo Robbiano. The Gröbner fan of an ideal. *Journal of Symbolic Computation*, 6:183–208, 1988.

[MS04] Ezra Miller and Bernd Sturmfels. *Combinatorial Commutative Algebra*, volume 227. Graduate Texts in Mathematics, Springer-Verlag, New York, 2004.

[Ray] Koushik Ray. String networks as tropical curves. arXiv:0804:1870.

[RGST03] Jügen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. First steps in tropical geometry. In G.L. Litvinov and V.P. Maslov, editors, *Proc. Conference on Idempotent Mathematics and Mathematical Physics*, pages 289–317, Vienna, 2003. American Mathematical Society, Contemporary Mathematics Vol 377.

[Sch86] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1986.

[Spe] David Speyer. Uniformizing tropical curves i: genus zero and one. arXiv:0711.2677.

[SS04] David Speyer and Bernd Sturmfels. The tropical Grassmannian. *Advances in Geometry*, 4:389–411, 2004.

[Sta80] R. Stanley. The number of faces of simplicial convex polytopes. *Advances Math.*, 35:236–238, 1980.

[Stu96] Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

[Vel08] Mauricio Velasco. Minimal free resolutions that are not supported by a cw-complex. *J. Algebra*, 319 no. 1:102–114, 2008.

[Vig] Magnus Vigeland. Tropical lines on smooth tropical surfaces. arXiv:0708.3847.

[Zie95] G.M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.