Lecture 6, Math 221: Multivariable calculus

Today's topic: Applications of triple integrals

But first: hints for ps1:

- In the first problem, switch the order of integration.
- In the third problem, evaluate the double integral of 1 in polar coordinates. Don't forget that

$$\cos^2\theta - \sin^2\theta = \cos 2\theta.$$

  Also, for future reference, keep in mind that

$$2\sin\theta\cos\theta = \sin 2\theta.$$

Today we'll see that, indeed, triple integration is useful. We already saw last time that we can use a triple integral to compute the volume of a solid (just integrate $f(x, y, z) = 1$).

Physics applications

The *moment* of an object of shape R about a coordinate plane measures its tendency to rotate about that plane. Note that there are three coordinate planes: the xy-plane, the xz-plane, and the yz-plane. If R is an object of mass m that occupies a single point, then its moment with respect to any one of the axes is simply m times its distance to the axis in question. More generally, we set

$$M_{xy} = \int\int\int_R z\rho(x, y, z)dV$$
$$M_{xz} = \int\int\int_R y\rho(x, y, z)dV$$
$$M_{yz} = \int\int\int_R x\rho(x, y, z)dV$$

where $\rho(x, y, z)$ is the density function of our object.

Moreover, the mass m of R is given by

$$m = \int\int\int_R \rho(x, y, z)dV.$$

The *center of mass*, or point inside the object that has no tendency to rotate, has coordinates

$$\overline{x} = \frac{M_{yz}}{m}, \overline{y} = \frac{M_{xz}}{m}, \overline{z} = \frac{M_{xy}}{m}.$$

Example: Find the center of mass of a solid of constant density bounded by the graphs of

$$x = y^2, \text{ and } x = z, z = 0, \text{ and } x = 1.$$

Let's compute the mass of our solid first. Its density function is a constant ρ. Accordingly,

$$m = \int\int\int_R \rho dV = \rho \int\int\int dV.$$

(We're just saying that mass is ρ times volume. No surprises here.) So we need to compute the volume V of R. Integrating first with respect to z looks most promising, so we have

$$V = \int\int\int dV = \int\int_\Omega \left[\int_0^x dz\right] dA$$

where Ω is the projection of R to the xy-plane. Integrating on the inside yields

$$V = \int\int_\Omega x dA.$$

On the other hand, it's not hard to see that

$$\Omega = \{(x, y)| -1 \le y \le 1, \text{ and } 1 \le x \le y^2\}.$$

Accordingly, we deduce that

$$\begin{aligned} V &= \int_{-1}^1 \int_{y^2}^1 x dx dy \\ &= \int_{-1}^1 \frac{1}{2}x^2\Big|_{x=y^2}^{x=1} 1 dy \\ &= \int_{-1}^1 \left(\frac{1}{2} - \frac{1}{2}y^4\right) dy \\ &= \frac{1}{2}y - \frac{1}{10}y^5\Big|_{-1}^1 \\ &= \frac{4}{5}. \end{aligned}$$

It follows that $m = \rho V = \frac{4}{5}\rho$.

Next we compute the moments of our object. To begin with, we have

$$M_{xz} = \int\int\int_R \rho y dV = 0,$$

because R is symmetric about the xz-plane. Computing the other two moments proceeds along the same lines we used to compute the mass of out object:

$$\begin{aligned} M_{xy} &= \int\int\int_R z\rho dV \\ &= \rho \int_{-1}^{1}\int_{y^2}^{1}\left[\int_0^x z dz\right] dxdy \\ &= \rho \int_{-1}^{1}\int_{y^2}^{1} \frac{1}{2}z^2\Big|_{z=0}^{z=x} dxdy \\ &= \rho \int_{-1}^{1}\int_{y^2}^{1} \frac{1}{2}x^2 dxdy \\ &= \rho \int_{-1}^{1} \frac{1}{6}x^3\Big|_{x=y^2}^{x=1} dy \\ &= \rho \int_{-1}^{1} \frac{1}{6} - \frac{1}{6}y^6 dy \\ &= \rho\left(\frac{1}{6}y - \frac{1}{42}y^7\right)\Big|_{-1}^{1} \\ &= \frac{2\rho}{7}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} M_{yz} &= \int\int\int_R x\rho dV \\ &= \rho \int_{-1}^{1}\int_{y^2}^{1}\left[\int_0^x x dz\right] dxdy \\ &= \rho \int_{-1}^{1}\int_{y^2}^{1} xz\Big|_{z=0}^{z=x} dxdy \\ &= \frac{4\rho}{7}. \end{aligned}$$

It follows that the center of mass of our object has coordinates

$$\overline{x} = \frac{\frac{4\rho}{7}}{\frac{4}{5}\rho} = \frac{5}{7}, \overline{y} = 0, \overline{z} = \frac{\frac{2\rho}{7}}{\frac{4}{5}\rho} = \frac{5}{14}.$$

More applications:

- The *total electric charge* on a solid object occupying a region R with charge density function $\sigma(x, yz)$ is

$$Q = \int \int \int_R \sigma(x, y, z) dV.$$

- The *joint density function* $f(x, y, z)$ of three random variables $X, Y,$ and Z measures the probability P that $X, Y,$ and Z jointly lie in a region R in space. It has the property that

$$P((X, Y, Z) \in R) = \int \int \int_R f(x, y, z) dV.$$