

## Lecture 7, Math 221: Multivariable calculus

### Today: Triple integrals in cylindrical and spherical coordinates

Thus far we've seen how to evaluate triple integrals in  $x, y$ , and  $z$ ; the next stop on our integration tour involves integrals over regions with radial symmetry in  $x$ , and  $z$ . We'll see that over such regions, life is made easier by using specially-adapted coordinates (much as we did in two variables, with polar coordinates) and applying an integral change-of-variable law.

#### Cylindrical coordinates

These are the direct analogue of polar coordinates for regions in 3-dimensional space. Any time our region  $R$  can be written in the form

$$R = \{(x, y, z) | (x, y) \in \Omega, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where  $\Omega$  may be represented in polar coordinates by

$$\Omega = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

we have the following change-of-variable formula:

$$\int \int \int_R f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta. \quad (0.1)$$

To see how the formula (0.1) arises, note first that

$$\begin{aligned} \int \int \int_R f(x, y, z) dV &= \int \int_{\Omega} \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \\ &= \int \int_{\Omega} G(x, y) dA \end{aligned}$$

where  $G(x, y) = \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz$ . But according to our change-of-variable formula for double integrals in polar coordinates,

$$\begin{aligned} \int \int_{\Omega} G(x, y) dA &= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} G(r \cos \theta, r \sin \theta) r dr d\theta \\ &= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta. \end{aligned}$$

Any time  $R$  is radially symmetric about the  $z$ -axis, but *not spherically symmetric*, life is probably made easier by using cylindrical coordinates.

**Example:** A solid  $T$  lies inside the cylinder  $x^2 + y^2 = 1$ , below the plane  $z = 4$ , and above the paraboloid  $z = 1 - x^2 - y^2$ . Given that the density at any point is proportional to its distance from the cylinder's axis, determine the mass of  $T$ .

Here  $T$  occupies a region  $R$  that is clearly radially symmetric about the  $z$ -axis. In terms of inequalities involving  $x, y$ , and  $z$ ,  $R$  is given by

$$R = \{(x, y, z) | 1 - x^2 - y^2 \leq z \leq 4, 0 \leq x^2 + y^2 \leq 1.\}$$

It's not hard to reexpress  $R$  in cylindrical coordinates; doing so, we obtain

$$R = \{(r, \theta, z) | 1 - r^2 \leq z \leq 4, r \leq 1, 0 \leq \theta \leq 2\pi\}.$$

Now let  $m$  denote the mass of  $T$ ; recall that

$$m = \int \int \int_R \rho(x, y, z) dV \tag{0.2}$$

where  $\rho(x, y, z)$  is the density function of  $T$ . Since density, in our case, is proportional to distance from the  $z$ -axis, we have

$$\rho(x, y, z) = K\sqrt{x^2 + y^2}$$

where  $K$  is a constant. Accordingly, our mass integral (0.2) becomes

$$\int \int \int_R K\sqrt{x^2 + y^2} dV,$$

which, according to (0.1), is in turn equal to

$$\int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr) r dz dr d\theta.$$

### Spherical coordinates

This a system of coordinates  $(\rho, \theta, \phi)$  that is useful whenever a region has "spherical" symmetry. (Think of spheres, and sections of spheres.) The website includes a link to a pictorial representation of spherical coordinates; they are related to the usual rectangular coordinates  $(x, y, z)$  by

$$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi.$$

Spherical coordinates are the appropriate coordinates to use whenever we are asked to evaluate a triple integral over a *spherical wedge*, i.e., any region realizable as

$$R = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}.$$

(A bit) more generally, spherical integrals are feasible to evaluate whenever  $R$  is of the form

$$R = \{(\rho, \theta, \phi) | \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}.$$

Triple integrals in spherical coordinates over spherical wedges obey the following change-of-variable law:

$$\int \int \int_R f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) (\rho^2 \sin \phi) d\rho d\theta d\phi. \quad (0.3)$$

**Example:** Compute  $\int \int \int_R e^{(x^2+y^2+z^2)^{3/2}} dV$  where

$$R = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$$

is the unit ball.

Our region is the unit sphere centered at the origin; accordingly, in spherical coordinates, we have

$$R = \{(\rho, \theta, \phi) | 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\},$$

and thus, by our change-of-variable formula (0.3),

$$\begin{aligned} \int \int \int_R e^{(x^2+y^2+z^2)^{3/2}} dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin \phi d\rho d\theta d\phi \\ &= \int_0^\pi \sin \phi d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3} d\rho \\ &= \frac{4}{3} \pi (e - 1). \end{aligned}$$