CONSTRAINT SATISFACTION PROBLEMS AND
HOMOMORPHISM VERSIONS OF ULTRAFILTER
AXIOMS

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ABSTRACT. We define filter-tolerant powers of finite relational structures, and consider the problem of retracting such structures to their diagonal. We show that for some structures, such retractions can be readily derived from the axioms of Zermelo and Fraenkel, while at the other extreme, there are structures for which the existence of such retractions is equivalent to the ultrafilter axiom.

1. Introduction

Let \( \mathcal{F} \) be a filter on a set \( I \), and \( \mathbb{A} \) a relational structure of some type \( \sigma \). The filter-tolerant power \( \mathbb{A}_I^\mathcal{F} \) is the structure defined as follows. The universe of \( \mathbb{A}_I^\mathcal{F} \) is the set \( A_I \) of all functions of \( I \) to the universe \( A \) of \( \mathbb{A} \), and for each \( R \in \sigma \) of arity \( k \), \( R(\mathbb{A}_I^\mathcal{F}) \subseteq (A_I)^k \) is the set of all \( k \)-tuples \( (f_1, \ldots, f_k) \) such that the set \( \{i \in I | (f_1(i), \ldots, f_k(i)) \in R(\mathbb{A})\} \) belongs to \( \mathcal{F} \).

Here, a filter is a collection of subsets of a set \( I \) which does not contain the empty set, is closed under finite intersections, and contains every superset of each of its members. A filter which is maximal with respect to inclusion is called an ultrafilter. Equivalently, a filter is an ultrafilter if and only if it contains precisely one of each pair of complementary subsets of \( I \). The ultrafilter axiom states that every filter extends to an ultrafilter. It is independent from the axioms of Zermelo and Fraenkel, but weaker than the axiom of choice.

The diagonal \( \Delta(\mathbb{A}_I^\mathcal{F}) \) of \( \mathbb{A}_I^\mathcal{F} \) is the substructure induced by the constant functions. It is naturally isomorphic to \( \mathbb{A} \). Therefore there exists a retraction of \( \mathbb{A}_I^\mathcal{F} \) to \( \Delta(\mathbb{A}_I^\mathcal{F}) \) if and only if there exists an idempotent homomorphism of \( \mathbb{A}_I^\mathcal{F} \) to \( \mathbb{A} \), that is, a homomorphism which maps each constant function to its constant value.

The existence of such homomorphisms is trivial when \( I \) is finite, but it may require additional hypotheses when \( I \) is infinite. It is relatively
easy to show that if $\mathcal{F}$ extends to an ultrafilter, then for every finite relational structure $A$, $A^I_{\mathcal{F}}$ retracts to $\Delta(A^I_{\mathcal{F}})$. We will show that for some structures, the converse holds: if $A^I_{\mathcal{F}}$ retracts to $\Delta(A^I_{\mathcal{F}})$, then $\mathcal{F}$ extends to an ultrafilter. In particular, in [2] it is shown that if $A$ is a complete graph on at least three vertices, then the retractions of $A^I_{\mathcal{F}}$ to $\Delta(A^I_{\mathcal{F}})$ correspond to the ultrafilters on $I$. The argument can be adapted to show that if $A^I_{\mathcal{F}}$ retracts to $\Delta(A^I_{\mathcal{F}})$, then $\mathcal{F}$ extends to an ultrafilter. At the other extreme, there are relational structures for which the existence of retractions of their filter-tolerant powers to their diagonals can be proved readily from the axioms of Zermelo and Fraenkel. In particular, if $A$ is a core structure (in the sense of [3]) of width one (in the sense of [1]), then for every filter $\mathcal{F}$, $A^I_{\mathcal{F}}$ retracts to $\Delta(A^I_{\mathcal{F}})$.

Thus the strength of the hypotheses “$A^I_{\mathcal{F}}$ retracts to $\Delta(A^I_{\mathcal{F}})$” ranges from the trivial to the ultrafilter axiom. Perhaps it would be interesting to connect the hierarchy of these hypotheses to the complexity hierarchy of constraint satisfaction problems. Note that the constraint satisfaction problem is NP-complete for complete graphs with at least three vertices, while it is polynomial for the core structures of width one. It is conceivable that the hypothesis is equivalent to the ultrafilter axiom for all structures for which the constraint satisfaction problem is known to be NP-complete. We do not know of structures that do not have width one for which the hypothesis is trivial.

2. Results

Proposition 1. If $\mathcal{F}$ is contained in an ultrafilter, then for every finite structure $A$, $A^I_{\mathcal{F}}$ retracts to $\Delta(A^I_{\mathcal{F}})$.

Proof. It suffices to show that there exists a idempotent homomorphism of $A^I_{\mathcal{F}}$ to $A$. Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{F}$. Then for every $f$ in $A^I$, there is a unique $x$ in $A$ such that $f^{-1}(x) \in \mathcal{U}$. We put $\phi(f) = x$. Then $\phi : A^I_{\mathcal{F}} \to A$ obviously maps constant functions to their constant value. We show that it preserves relations. Let $R \in \sigma$ be a relation of arity $k$, and $(f_1, \ldots, f_k) \in R(A^I_{\mathcal{F}})$. Then

$$S = \{ i \in I | (f_1(i), \ldots, f_k(i)) \in R(A) \} \in \mathcal{F} \subseteq \mathcal{U}.$$ 

Therefore the set $S \cap \left( \bigcap_{j=1}^k f_j^{-1}(\phi(f_j)) \right)$ is in $\mathcal{U}$, hence it is not empty. For $i \in S \cap \left( \bigcap_{j=1}^k f_j^{-1}(\phi(f_j)) \right)$, we have

$$\phi(f_1), \ldots, \phi(f_k)) = (f_1(i), \ldots, f_k(i)) \in R(A).$$ 

$\square$
Let $K_n$ denote the complete graph on $n$ vertices, that is, the structure with universe $\{0,\ldots,n-1\}$ and the binary adjacency relation $\neq$.

**Proposition 2.** If $(K_n)^I_\mathcal{F}$ retracts to $\Delta((K_n)^I_\mathcal{F})$ for some finite $n \geq 3$, then $\mathcal{F}$ is contained in an ultrafilter.

**Proof.** Let $\phi : (K_n)^I_\mathcal{F} \rightarrow K_n$ be an idempotent homomorphism. For $k \in \{0,\ldots,n-1\}$, we write $\text{id}_k$ for the constant function with constant value $k$. If $n > 3$, every element of $(K_3)^I_\mathcal{F} \subseteq (K_n)^I_\mathcal{F}$ is adjacent to $\text{id}_k$ for all $k \geq 3$. Therefore the restriction of $\phi$ to $(K_3)^I_\mathcal{F}$ is an idempotent homomorphism to $K_3$. Therefore we can assume that $n = 3$.

For $X \subseteq I$, we write $\mathbb{1}_X$ for the characteristic map of $X$, that is, $\mathbb{1}_X(i) = 1$ if $i \in X$ and $\mathbb{1}_X(i) = 0$ otherwise. Then $\mathbb{1}_X$ is adjacent to $\text{id}_2$, therefore $\phi(\mathbb{1}_X) \in \{0,1\}$ for all $X \subseteq I$. Put

$$\mathcal{U} = \{X|\phi(\mathbb{1}_X) = 1\}.$$ We will show that $\mathcal{U}$ is an ultrafilter containing $\mathcal{F}$.

For $F \in \mathcal{F}$, $\mathbb{1}_F$ is adjacent to $\text{id}_0$. Since $\phi(\text{id}_0) = 0$, we have $\phi(\mathbb{1}_F) = 1$. Thus, $\mathcal{F} \subseteq \mathcal{U}$. Also, for any $X \subseteq I$, $\mathbb{1}_X$, $\mathbb{1}_{X^c}$ and $\text{id}_2$ are mutually adjacent. (Where $X^c$ denote the complement of $X$.) Since $\phi(\text{id}_2) = 2$, we must have $\{\phi(\mathbb{1}_X),\phi(\mathbb{1}_{X^c})\} = \{0,1\}$, that is, $\mathcal{U}$ contains precisely one of $X$ and $X^c$.

For $X \in \mathcal{U}$, define $f_X, g_X : I \rightarrow \{1,2\}$ by

$$(f_X(i), g_X(i)) = \begin{cases} (2,1) & \text{if } i \in X, \\ (1,2) & \text{otherwise.} \end{cases}$$

Then $f_X$ is adjacent to $\text{id}_0$ and $\mathbb{1}_X$, thus $\phi(f_X) = 2$. Since $g_X$ is adjacent to $\text{id}_0$ and $f_X$, we then have $\phi(g_X) = 1$. Now for any $Y \subseteq I$ containing $X$, $\mathbb{1}_Y$ is adjacent to $g_X$, thus $\phi(\mathbb{1}_Y) = 0$ and $\phi(\mathbb{1}_{Y^c}) = 1$. This shows that if $Y$ contains $X \in \mathcal{U}$, then $Y \in \mathcal{U}$.

For $X, Y \in \mathcal{U}$, define $f_{X \cap Y}, f_{X \setminus Y}, f_X : I \rightarrow \{0,1,2\}$ by

$$(f_{X \cap Y}(i), f_{X \setminus Y}(i), f_X(i)) = \begin{cases} (0,1,2) & \text{if } i \in X \cap Y, \\ (2,0,1) & \text{if } i \in X \setminus Y, \\ (1,2,0) & \text{if } i \in X^c. \end{cases}$$

Then $f_{X \cap Y}$, $f_{X \setminus Y}$ and $f_X$ are mutually adjacent, so $\{\phi(f_{X \cap Y}), \phi(f_{X \setminus Y}), \phi(f_X)\} = \{0,1,2\}$. Since $f_X$ is adjacent to $\mathbb{1}_X$ and $\phi(\mathbb{1}_X) = 0$, we have $\phi(f_X) \neq 0$. Similarly, $X \setminus Y \subseteq \overline{Y}$ whence $\phi(\mathbb{1}_{X \setminus Y}) = 0$, and $\mathbb{1}_{X \setminus Y}$ is adjacent to $f_{X \setminus Y}$, so that $f_{X \setminus Y} \neq 0$. Therefore $\phi(f_{X \cap Y}) = 0$. Since $\mathbb{1}_{X \setminus Y}$ is adjacent to $f_{X \setminus Y}$, we then have $\phi(\mathbb{1}_{X \setminus Y}) = 1$, that is, $X \cap Y \in \mathcal{U}$. This shows that $\mathcal{U}$ is an ultrafilter. \qed
The proof method readily adapts to many graphs, but apparently not to bipartite graphs. The existence of a retraction of \((K_2)^I\) to \(\Delta((K_2)^I)\) implies that \(\mathcal{F}\) extends to a family \(\mathcal{U}\) which contains precisely one of each pair of complementary subset of \(I\), and has the property that if \(X \in \mathcal{U}\) and \((X \cap Y) \cup (\overline{X} \cap \overline{Y}) \in \mathcal{F}\), then \(Y \in \mathcal{U}\). Adding relations or considering other relational structures on \(\{0, 1\}\) can alter the properties that extensions of \(\mathcal{F}\) might satisfy; in some cases it can be shown that \(\mathcal{F}\) indeed extends to an ultrafilter. Perhaps in the case of relations on \(\{0, 1\}\), a complete hierarchy of the corresponding hypotheses can be obtained.

**Proposition 3.** Let \(\mathbb{A}\) be a core structure of width one. Then for every filter \(\mathcal{F}\), \(A^I_{\mathcal{F}}\) retracts to \(\Delta(A^I_{\mathcal{F}})\).

**Proof.** Following [1], a structure \(\mathbb{A}\) has width one if and only if there is a homomorphism \(\psi : \mathcal{P}(\mathbb{A}) \to \mathbb{A}\), where \(\mathcal{P}(\mathbb{A})\) is the structure defined as follows. The universe of \(\mathcal{P}(\mathbb{A})\) is the set of nonempty subsets of the universe \(A\) of \(\mathbb{A}\). For \(R \in \sigma\) of arity \(k\), \(R(\mathcal{P}(\mathbb{A}))\) consists of the \(k\)-tuples \((S_1, \ldots, S_k)\) such that \(\text{pr}_i(R(\mathbb{A})) \cap (\Pi_{i=1}^k S_i) = S_i\) for \(i = 1, \ldots, k\). (Where \(\text{pr}_i\) is the \(i\)-th projection.)

There is a natural copy of \(\mathbb{A}\) in \(\mathcal{P}(\mathbb{A})\) induced by the singletons. The structure \(\mathbb{A}\) is called a core if it has no proper retract. By a result in [3], if \(\mathbb{A}\) is a core with width 1, then \(\mathbb{A}\) is rigid in the sense that it has no non-trivial endomorphism. Therefore \(\psi : \mathcal{P}(\mathbb{A}) \to \mathbb{A}\) satisfies \(\psi(\{x\}) = x\) for all \(x \in A\).

Now for \(f\) in the universe of \(A^I_{\mathcal{F}}\), put

\[
S(f) = \{S \subseteq A : f^{-1}(S) \in \mathcal{F}\}.
\]

In particular, \(A \in S(f)\), \(\emptyset \not\in S(f)\), and \(S(f)\) is closed under intersections. Let \(\phi(f)\) be the smallest member of \(S(f)\). We show that \(\phi : A^I_{\mathcal{F}} \to \mathcal{P}(\mathbb{A})\) is a homomorphism.

Let \(R \in \sigma\) be a relation of arity \(k\), and \((f_1, \ldots, f_k)\) an element of \(R(A^I_{\mathcal{F}})\). Put

\[
S = \{i \in I : (f_1(i), \ldots, f_k(i)) \in R(\mathbb{A})\}.
\]

Then \(S\) is in \(\mathcal{F}\), hence \(T = S \cap (\bigcap_{i=1}^k f_i^{-1}(\phi(f_i))) \in \mathcal{F}\). For \(i = 1, \ldots, k\), we have \(\{f_i(t) : t \in T\} \subseteq \phi(f_i)\), but by minimality of \(\phi(f_i)\), the inclusion cannot be strict since \(f_i^{-1}(\{f_i(t) : t \in T\}) \in \mathcal{F}\). Thus every \(x \in \phi(f_i)\) is the \(i\)-th coordinate of some \((f_1(t), \ldots, f_k(t))\) in \(R(\mathbb{A})\), with \(t \in T\) and \(f_j(t) \in \phi(f_j)\) for \(j = 1, \ldots, k\). In other words, \(\text{pr}_i(R(\mathbb{A}) \cap (\Pi_{i=1}^k \phi(f_i))) = \phi(f_i)\) for \(i = 1, \ldots, k\), whence \((\phi(f_1), \ldots, \phi(f_k)) \in R(\mathcal{P}(\mathbb{A}))\). This shows that \(\phi\) is a homomorphism.
Composing with $\psi: \mathcal{P}(A) \to A$, we get $\psi \circ \phi: \mathbb{A}_F^I \to A$, mapping constant functions to their constant image. Composing with the natural isomorphism between $A$ and $\Delta(A_F^I)$ yields the desired retraction. □

3. Further comments

Let $\sim$ be the equivalence on $\mathbb{A}_F^I$ defined by $f \sim g$ if $\{i \in I | f(i) = g(i)\} \in \mathcal{F}$. It is easy to see that $\mathbb{A}_F^I$ is a “lexicographic sum” of the equivalence classes of $\sim$: if $(f_1, \ldots, f_k) \in R$ and $g_j \sim f_j$, $j = 1, \ldots, k$, then $(g_1, \ldots, g_k) \in R$. Any homomorphism of $\mathbb{A}_F^I/\sim$ to $A$ can be composed with the quotient map of $\mathbb{A}_F^I$ to $\mathbb{A}_F^I/\sim$. Conversely, when $A$ is finite, any homomorphism $\phi: \mathbb{A}_F^I \to A$ allows to define a homomorphism $\psi: \mathbb{A}_F^I/\sim \to A$ as follows: given an ordering of the universe of $A$, define $\psi(x/\sim)$ to be the smallest $a$ such that there exists $y \in x/\sim$ with $\phi(y) = a$. Thus the existence of a homomorphism of $\mathbb{A}_F^I$ to $A$ is equivalent to the existence of a homomorphism of $\mathbb{A}_F^I/\sim$ to $A$. When $\mathcal{F}$ is an ultrafilter, $\mathbb{A}_F^I/\sim$ is the standard ultrapower construction.

When $A$ is infinite, things are different. For instance, let $A$ be the one-way infinite path, that is, the digraph on the universe $\mathbb{N}$ with the successor relation $\{(i, i + 1)|i \in \mathbb{N}\}$. Let $\mathcal{F}$ be the Fréchet filter on $\mathbb{N}$, that is, the family of cofinite subsets of $\mathbb{N}$. Then $\mathbb{A}_F^\mathbb{N}$ contains the infinite descending path $f_0, f_1, f_2, \ldots$ defined by $f_i(j) = \max\{j - i, 0\}$. Therefore there is no homomorphism of $\mathbb{A}_F^\mathbb{N}$ to $A$. However, if $A$ is the two-way infinite path (the digraph on $\mathbb{Z}$ with the successor relation), then the existence of a homomorphism of $\mathbb{A}_F^\mathbb{N}$ to $A$ follows from the axiom of choice. Indeed, two elements $f, g$ of $\mathbb{A}_F^\mathbb{N}$ are in the same connected component if there exists $j \in \mathbb{Z}$ such that $\{i \in \mathbb{N}|f(i) - g(i) = j\} \in \mathcal{F}$. A homomorphism $\phi: \mathbb{A}_F^\mathbb{N} \to A$ can then be defined by selecting a representative in each connected component of $\mathbb{A}_F^\mathbb{N}$. If $f$ is an element of $\mathbb{A}_F^\mathbb{N}$ and $g$ is the representative of the connected component of $f$, then $\phi(f)$ is defined as the unique $j \in \mathbb{Z}$ such that $\{i \in \mathbb{N}|f(i) - g(i) = j\} \in \mathcal{F}$.

We define the $1$-tolerant power $\mathbb{A}_I^I$ as follows. The universe of $\mathbb{A}_I^I$ is $\mathbb{A}^I$, and for each $R \in \sigma$ of arity $k$, $R(\mathbb{A}_I^I) \subseteq (\mathbb{A}^I)^k$ is the set of all $k$-tuples $(f_1, \ldots, f_k)$ such that $(f_1(i), \ldots, f_k(i)) \in R(A)$ for all except at most element $i$ of $I$. In [3], it is shown that there exists a homomorphism of some $\mathbb{A}_I^I$ to $A$ for some finite $I$ if and only if $A$ has finite duality. When $I$ is infinite, there exists a homomorphism of $\mathbb{A}_I^I$ to $\mathbb{A}_I^I$, where $\mathcal{F}$ is the Fréchet filter on $I$. Thus if $\mathcal{F}$ extends to an ultrafilter, then there exists a homomorphism of $\mathbb{A}_I^I$ to $A$. In all the case where we show that a retraction of $\mathbb{A}_I^I$ to $\Delta(\mathbb{A}_F^I)$ implies that $\mathcal{F}$ extends to
an ultrafilter, it can be seen that the existence of a retraction of $A_1^I$ to $\Delta(A_1^I)$ already implies the same conclusion.

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**References**


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