GENERALISED MYCIELSKI GRAPHS AS TOPOLOGICAL CLIQUES

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Abstract. We prove that the coindex of the box complex $B(H)$ of a graph $H$ can be measured by the generalised Mycielski graphs which admit a homomorphism to it. As a consequence, we exhibit for every graph $H$ a system of linear equations solvable in polynomial time, with the following properties: if the system has no solutions, then $\text{coind}(B(H)) + 2 \leq 3$; if the system has solutions, then $\chi(H) \geq 4$.

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1. Introduction

For any integer $k \geq 2$ and real number $\epsilon \in (0, 1)$, the Borsuk graph $B_{k,\epsilon}$ is the graph whose vertices are the points of the $(k - 2)$-sphere $S_{k-2}$, and whose edges join pairs of points $X, Y$ that are “almost antipodal” in the sense that the norm of $X - Y$ is at least $2 - \epsilon$. In [4], Erdős and Hajnal used the Borsuk-Ulam theorem to prove that the chromatic number of $B_{k,\epsilon}$ is $k$. In fact, they proved that the statement $\chi(B_{k,\epsilon}) = k$ is equivalent to the Borsuk-Ulam theorem. For some years this result remained a curiosity involving infinite graphs. Then Lovász [5] devised complexes that allow the use of the Borsuk-Ulam Theorem to find lower bounds on chromatic numbers of finite graphs, and used this method to prove the Kneser conjecture on the chromatic number of the Kneser graphs.

Lovász' method inspired many adaptations and developments, giving rise to the field of “topological lower bounds” on the chromatic number of a graph. We will focus on a bound in terms of “coindices of box complexes”, specifically

$$\chi(H) \geq \text{coind}(B(H)) + 2.$$ 

The relevant definitions of the coindex $\text{coind}(B(H))$ of the box complex $B(H)$ of $H$ are well detailed in [6, 7, 8]. However, our intent is to avoid the topological setting. We will use the following result of Simonyi and Tardos.

Theorem 1 ([8]). For any graph $H$, $\text{coind}(B(H)) + 2$ is the largest $k$ such that there exist $\epsilon > 0$ for which $B_{k,\epsilon}$ admits a homomorphism (that is, an edge-preserving map) to $H$.

This result indeed allows us to restrict our discussion to the field of graphs and homomorphisms: We can alternatively define $\text{coind}(B(H)) + 2$ as the largest $k$ such that there exist an $\epsilon > 0$ for which $B_{k,\epsilon}$ admits a homomorphism to $H$. Thus the Borsuk graphs may be viewed as “topological cliques”, with $\text{coind}(B(H)) + 2$ as the corresponding “topological clique number”. This viewpoint yields an economy of approach.

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definitions, but is not necessarily practical for computational purposes. Indeed in many cases a knowledge of simplicial complexes and topological tricks is needed to compute \( \text{coind}(B(H)) + 2 \) and effectively bound \( \chi(H) \).

Dochtermann and Schultz [3] found finite ("spherical") graphs that can be used as topological cliques and play the role of the Borsuk graphs in Theorem 1. In this note we show that the generalised Mycielski graphs can also be used in the the same role. We do not know whether this alternative presentation yields effective computations in the general case. However, for low values of \( \text{coind}(B(H)) \), our definition indeed leads to practical calculations that can be shown to be conclusive in some cases. This leads to open questions as to whether our method indeed decides whether an input graph \( H \) satisfies \( \text{coind}(B(H)) + 2 \leq 3 \) in polynomial time.

2. Generalised Mycielski graphs

The categorical product of two graphs \( G \) and \( G' \) is the graph \( G \times G' \) defined by
\[
V(G \times G') = V(G) \times V(G'),
\]
\[
E(G \times G') = \{[(u, u'), (v, v')]: [u, v] \in E(G) \text{ and } [u', v'] \in E(G')\}.
\]
We sometimes use directed graphs as factors, and view undirected graphs as symmetric directed graphs. In this case, all square brackets representing edges in the above definition should be replaced by parentheses representing arcs.

For \( n \in \mathbb{N}^* \), let \( P_n \) denote the path with vertices \( 0, 1, \ldots, n \) linked consecutively, with a loop at 0. For a graph \( G \), the \( n \)-th cone \( M_n(G) \) (or \( n \)-th generalised Mycielskian) over \( G \) is the graph \( (G \times P_n)/\sim_n \), where \( \sim_n \) is the equivalence which identifies all vertices whose second coordinate is \( n \). This construction allows us to define classes of "generalised Mycielski graphs". We proceed inductively: Let \( \mathcal{K}_2 = \{K_2\} \), and for \( k \geq 3 \), put
\[
\mathcal{K}_k = \{M_n(G): G \in \mathcal{K}_{k-1}, n \in \mathbb{N}^*\}.
\]
Csorba [1, 2] proved that for any graph \( H \) and integer \( n \), the geometric realisation of \( B(M_n(H)) \) is \( \mathbb{Z}_2 \)-homotopy equivalent to the geometric realisation of the suspension of \( B(H) \). In particular this implies that \( \text{coind}(B(G)) + 2 = \chi(G) = k \) for every \( G \in \mathcal{K}_k \).

**Lemma 2.** For every Borsuk graph \( B_{k, \epsilon} \), there exists a graph \( G \) in \( \mathcal{K}_k \) such that \( G \) admits a homomorphism to \( B_{k, \epsilon} \).

**Proof.** \( \mathcal{K}_2 = \{K_2\} \), and \( B_{2, \epsilon} = K_2 \) for every \( \epsilon > 0 \). Suppose that \( G \in \mathcal{K}_{k-1} \) admits a homomorphism to \( B_{k-1, \epsilon/2} \). Put \( n = \lceil \pi/\epsilon \rceil \). We will show that \( M_n(G) \) admits a homomorphism to \( B_{k, \epsilon} \). We identify the vertex set of \( B_{k-1, \epsilon/2} \) with the equator of that of \( B_{k, \epsilon} \). Hence \( [u, v] \in E(B_{k-1, \epsilon/2}) \) implies \( [u, v] \in E(B_{k, \epsilon}) \). Let \( p_N, p_S \) respectively be the north and south poles of \( B_{k, \epsilon} \). Let \( \phi : G \rightarrow B_{k-1, \epsilon/2} \) be a homomorphism. For every \( u \in V(G) \), let \( \phi(u) = u_{N,0}, u_{N,1}, \ldots, u_{N,n} = p_N \) be equally spaced points on the quarter of the great circle joining \( \phi(u) \) and \( p_N \). Similarly let \( \phi(u) = u_{S,0}, u_{S,1}, \ldots, u_{S,n} = p_S \) be equally spaced points on the quarter of the great circle joining \( \phi(u) \) and \( p_S \). Define \( \psi : M_n(G) \rightarrow B_{k, \epsilon} \) by
\[
\psi(u, i) = \begin{cases} 
  u_{N,i} & \text{if } i \text{ is even}, \\
  u_{S,i} & \text{if } i \text{ is odd}.
\end{cases}
\]
Note that \( \psi(u, n) = p_n \) or \( p_S \) according to whether \( n \) is even or odd, hence \( \psi \) is well defined. Also, \( \psi \) extends \( \phi \). For \( [u, v] \in E(G) \) and \( i < m \), we have
\[
||\psi(u, i) + \psi(v, i + 1)|| = ||u_{N,i} + v_{S,i+1}|| = ||u_{N,0} + v_{S,1}||
\geq ||u_{N,0} + v_{S,0}|| - ||v_{S,0} - v_{S,1}|| > 2 - \epsilon.
\]
Therefore \( \psi \) is a homomorphism.

**Corollary 3.** For any graph \( H \), \( \text{coind}(B(H)) + 2 \) is the largest \( k \) such that there exists a \( G \) in \( K_k \) admitting a homomorphism to \( H \).

**Proof.** Let \( k = \text{coind}(B(H)) + 2 \). By Theorem 1 and Lemma 2, there exist a number \( \epsilon > 0 \) and a graph \( G \in K_k \) such that there are homomorphisms of \( B_{k,\epsilon} \) to \( H \) and of \( G \) to \( B_{k,\epsilon} \). The composition of these is a homomorphism of \( G \) to \( H \). On the other hand, it is well known that if \( G \) admits a homomorphism to \( H \), then \( \text{coind}(B(G)) \leq \text{coind}(B(H)) \).

For an integer \( k \geq 2 \), none of the original definition, Theorem 1 and Corollary 3 imply that the problem of determining whether an input graph \( H \) satisfies \( \text{coind}(B(H)) + 2 \leq k \) is even decidable. For \( k = 2 \) the problem is equivalent to that of determining whether \( H \) is bipartite, which admits an efficient solution. For the remainder of the paper, we will focus on the case \( k = 3 \) and present an approach derived from Corollary 3.

### 3. Signatures of odd cycles

For two graphs \( G \) and \( H \), the exponential graph \( H^G \) has for vertices all functions \( f : V(G) \to V(H) \), and for edges all pairs \([f, g]\) of functions such that for every \([u, v] \in E(G)\), \([f(u), g(u)] \in E(H)\). In particular \( f \) is a homomorphism of \( G \) to \( H \) if and only if \( f \) is a loop in \( H^G \). A homomorphism of \( G \times G' \) to \( H \) corresponds to a homomorphism of \( G' \) to \( H^G \). In particular, with \( G' = \mathbb{P}_m \), we have the following.

**Remark 4.** A homomorphism of \( M_m(G) \) to \( H \) corresponds to a \( m \)-path in \( H^G \) from a loop to a constant map.

A graph \( H \) satisfies \( \text{coind}(B(H)) + 2 \geq 2 \) if and only if \( H \) contains an edge, and \( \text{coind}(B(H)) + 2 \geq 3 \) if and only if \( H \) contains an odd cycle. We have \( \text{coind}(B(H)) + 2 \geq 4 \) if and only if for some odd cycle \( C \), the connected component of a constant map in \( H^C \) contains loops.

Fix an odd cycle \( C \) with vertex-set \( \mathbb{Z}_{2n+1} = \{0, 1, \ldots, 2n\} \). We will devise group-theoretic signatures \( \sigma \) such that if a homomorphism \( f : C \to H \) is in the connected component of a constant map in \( H^C \), then \( \sigma(f) \) is the identity.

For a graph \( H \), let \( A(H) \) denote the set of its arcs. That is, for \([u, v] \in E(H)\), \( A(H) \) contains the two arcs \((u, v)\) and \((v, u)\). Let \( \mathcal{F}(A(H)) = \mathbb{Z}^{A(H)} \) and \( \mathcal{F}^*(A(H)) \) respectively be the free abelian group and the free group generated by the elements of \( A(H) \). For an arc \((f, g)\) of \( H^C \), we define its \( \mathcal{F}(A(H)) \)-signature \( \sigma_{\mathcal{F}(A(H))}(f, g) \) as
\[
\sigma_{\mathcal{F}(A(H))}(f, g) = \sum_{i=0}^{2n} (f(2i), g(2i+1)) - (f(2i+2), g(2i+1))
\]
and its $F^*(A(H))$-signature $\sigma_{F^*(A(H))}(f,g)$ as
\[
\sigma_{F^*(A(H))}(f,g) = \Pi_{i=0}^{2n}(f(2i), g(2i + 1)) \cdot (f(2i + 2), g(2i + 1))^{-1}
\]
(where the indices are taken modulo $2n + 1$, and the product is developed left to right).

Define $\alpha_0 : A(H) \to F(A(H))$ by $\alpha_0(u,v) = -(v,u)$. Then $\alpha_0$ extends to an order 2 automorphism $\alpha$ of $F(A(H))$. For every arc $(f,g)$ of $H^C$, $\sigma_{F(A(H))}(g,f) = \alpha(\sigma_{F(A(H))}(f,g))$. Similarly, $\alpha_0^* : A(H) \to F^*(A(H))$ defined by $\alpha_0^*(u,v) = (v,u)^{-1}$ extends to an automorphism $\alpha^*$ of $F^*(A(H))$. For every arc $(f,g)$ of $H^C$, $\sigma_{F^*(A(H))}(g,f)$ is the conjugate of $\alpha^*(\sigma_{F^*(A(H))}(f,g))$ obtained by interchanging the first $2n + 1$ terms of the product with the last $2n + 1$ terms.

We define a congruence $\theta$ on $F(A(H))$ as follows: If $u,v,w,x$ is a 4-cycle of $H$, we put
\[
(u,v) - (w,v) \theta (u,x) - (w,x),
\]
that is,
\[
(u,v) - (w,v) + (w,x) - (u,x) \theta 0_{F(A(H))}.
\]
Similarly, $\theta^*$ is defined on $F^*(A(H))$ as follows: If $uv,w,x$ is a 4-cycle of $H$, we put
\[
(u,v) \cdot (w,v)^{-1} \theta^* (u,x) \cdot (w,x)^{-1},
\]
that is,
\[
(u,v) \cdot (w,v)^{-1} \cdot (w,x) \cdot (u,x)^{-1} \theta^* 1_{F^*(A(H))}.
\]
Put $G(H) = F(A(H))/\theta$. For $W,W' \in F(A(H))$, $W\theta W'$ implies $\alpha(W)\theta \alpha(W')$. Hence $\alpha$ induces a well-defined automorphism of $G(H)$, which we will also call $\alpha$. Similarly, $\alpha^*$ induces a well-defined automorphism of $G^*(H) := F^*(A(H))/\theta^*$, which we will also call $\alpha^*$.

If $(f,g),(f,g')$ are arcs of $H^C$, then $\sigma_{F(A(H))}(f,g)\theta \sigma_{F(A(H))}(f,g')$. Therefore if $f$ is a non-isolated vertex of $H^C$, then $\sigma_{F(A(H))}(f,g)/\theta$ is independent of the choice of the neighbour $g$ of $f$. We define $\sigma_{G(H)}(f) = \sigma_{F(A(H))}(f,g)/\theta$ for any neighbour $g$ of $f$. Similarly, $\sigma_{G^*(H)}(f) = \sigma_{F^*(A(H))}(f,g)/\theta^*$ for any neighbour $g$ of $f$.

**Proposition 5.** If $f$ belong to the connected component of a constant in $H^C$, then $\sigma_{G(H)}(f) = 0_{G(H)}$ and $\sigma_{G^*(H)}(f) = 1_{G^*(H)}$.

**Proof.** Note that $G(H) = G^*(H)/\gamma$, where $\gamma$ is the commutator of $G^*(H)$. Therefore it suffices to prove that $\sigma_{G^*(H)}(f) = 1_{G^*(H)}$ whenever $f$ belong to the connected component of a constant in $H^C$.

If $f$ is a constant map, then $\sigma_{F^*(A(H))}(f,g) = 1_{F^*(A(H))}$ for any neighbour $g$ of $f$, hence $\sigma_{G^*(H)}(f) = 1_{G^*(H)}$. If $f$ is any element of $H^C$ such that $\sigma_{G^*(H)}(f) = 1_{G^*(H)}$, then for any neighbour $g$ of $f$,
\[
\sigma_{G^*(H)}(g) = \sigma_{F^*(A(H))}(g,f)/\theta = (W \cdot \alpha^*(\sigma_{F^*(A(H))}(f,g)) \cdot W^{-1})/\theta = 1_{G^*(H)}.
\]
By connectivity, this implies that $\sigma_{G^*(H)}(f)$ is identically $1_{G^*(H)}$ on the connected component of a constant in $H^C$.

The converse does not hold in general (see Section 4).
Proposition 6. Let $H$ be a connected graph. Consider the following system of equations in the variables $X_{u,v}, (u,v) \in A(H)$ and $N$.

(1) $\sum_{v \in N_H(u)} (X_{u,v} - X_{v,u}) = 0$ for all $u \in V(H)$,

(2) $\sum_{(u,v) \in A(H)} X_{u,v} - 2N = 1$,

(3) $\left( \sum_{(u,v) \in A(H)} (X_{u,v} - X_{v,u}) \cdot (u,v) \right) / \theta = 0_{G(H)}$.

This system admits integer solutions if and only if there exists an odd cycle $C$ and a homomorphism $f : C \to H$ such that $\sigma_{G(H)}(f) = 0_{G(H)}$.

Proof. Let $C$ be an odd cycle (with $V(C) = \mathbb{Z}_{2n+1}$) and $f : C \to H$ a homomorphism. Put $N = n$ and

$X_{u,v} = |\{i \in \mathbb{Z}_{2n+1} : f(i) = u, f(i+1) = v\}|$

for every arc $(u,v) \in A(H)$. Then the set of values $X_{u,v}, (u,v) \in A(H)$ and $N$ satisfy the flow constraints (1) and the parity constraint (2). We have

$\sigma_{G(H)}(f) = \left( \sum_{(u,v) \in A(H)} (X_{u,v} - X_{v,u}) \cdot (u,v) \right) / \theta,$

and this value is $0_{G(H)}$ if and only if $X_{u,v}, (u,v) \in A(H)$ and $N$ are solutions to the system.

Conversely, let $X_{u,v}, (u,v) \in A(H)$ and $N$ be an integer solution to the system. We first modify the solution by subtracting $\min\{X_{u,v}, X_{v,u}\}$ from both $X_{u,v}$ and $X_{v,u}$ for every edge $[u,v]$ of $H$ and subtracting the sum of these minima from $N$. We now have a non-negative integer solution. We then add 1 to every variable $X_{u,v}, (u,v) \in A(H)$, and $|E(H)|$ to $N$. This yields a positive integer solution with connected support. Let $G$ be the multigraph with $V(G) = V(H)$ and $X_{u,v}$ parallel arcs connecting $u$ to $v$ for every $(u,v) \in A(H)$. Then $G$ is connected Eulerian, and an Euler closed trail in $G$ corresponds to a homomorphism $f : C \to H$ with $|V(C)| = 2N + 1$ and

$\sigma_{G(H)}(f) = \left( \sum_{(u,v) \in A(H)} (X_{u,v} - X_{v,u}) \cdot (u,v) \right) / \theta = 0_{G(H)}$.

Corollary 7. Let $H$ be graph such that the system of equations (1), (2), (3) of Proposition 6 has no integer solutions. Then $\text{coin}(B(H)) + 2 \leq 3$.

Proposition 8. Let $H$ be graph such that the system of equations (1), (2), (3) of Proposition 6 has integer solutions. Then $\chi(H) \geq 4$.

Proof. We will prove more generally that if there exists a homomorphism $\phi : H \to H'$ and the system of equations (1), (2), (3) of Proposition 6 has integer solutions on $H$, then it has integer solutions on $H'$.
Then for \( u \) therefo
er the flow constraint (1) is satisfied. Also, equations (1), (2), (3) of Proposition 6. For 
therefore the parity constraint (2) is satisfied. Finally since 
we again call \( \hat{\phi} \) therefore the signature constraint (3) is satisfied. This proves that 
\[ \hat{\phi} : A(H) \to A(H') \] defined by \( \hat{\phi}(u, v) = (\phi(u), \phi(v)) \) extends to a
group homomorphism of \( F(A(H)) \) to \( F(A(H')) \), which we also denote \( \hat{\phi} \). For 
\((u, v), (w, v), (w, x), (u, x) \in A(H)\), we have 
\[ \hat{\phi}(u, v)- (w, v) + (w, x) - (u, x) = (\phi(u), \phi(v)) - (\phi(w), \phi(v)) + (\phi(w), \phi(x)) - (\phi(u), \phi(x)) \]
which is congruent to \( 0_{F(A(H'))} \) (or even equal to \( 0_{F(A(H'))} \) if \( \phi(u), \phi(v), \phi(w), \phi(x) \)
are not all distinct). Therefore \( \hat{\phi} \) induces a homomorphism of \( G(H) \) to \( G(H') \), which 
we again call \( \hat{\phi} \).

Now suppose that \( X_{u,v}, (u, v) \in A(H) \) and \( N \) are solutions to the system of 
equations (1), (2), (3) of Proposition 6. For \((u', v') \in A(H')\), put 
\[ X_{u',v'} = \sum \{ X_{u,v} : \phi(u) = u' \text{ and } \phi(v) = v' \}. \]
Then for \( u' \in V(H') \) we have 
\[ \sum_{v' \in N_{H'}(u')} (X_{u',v'} - X_{v',u'}) = \sum_{u \in \phi^{-1}(u')} \sum_{v \in \phi_{H'}(u)} (X_{u,v} - X_{v,u}) = 0, \]
therefore the flow constraint (1) is satisfied. Also, 
\[ \sum_{(u', v') \in A(H')} X_{u',v'} - 2N = \sum_{(u, v) \in A(H)} X_{u,v} - 2N = 1, \]
therefore the parity constraint (2) is satisfied. Finally since 
\[ \left( \sum_{(u, v) \in A(H)} (X_{u,v} - X_{v,u}) \cdot (u, v) \right) \mod 0_{G(H)} \]
and \( \hat{\phi} \) is a group homomorphism, we have 
\[ \left( \sum_{(u, v) \in A(H)} (X_{u,v} - X_{v,u}) \cdot (\phi(u), \phi(v)) \right) \mod 0_{G(H')} \]
Regrouping preimages we get 
\[ \left( \sum_{(u', v') \in A(H')} (X_{u',v'} - X_{v',u'}) \cdot (u', v') \right) \mod 0_{G(H')}, \]
therefore the signature constraint (3) is satisfied. This proves that \( X_{u',v'}, (u', v') \in 
A(H') \) and \( N \) are integer solutions to the system.
Now the graph \( H' = K_3 \) has no 4-cycles, therefore \( G(K_3) = F(A(K_3)) = Z^{A(K_3)} \).
Therefore the signature constraint (3) 
\[ \left( \sum_{(u', v') \in A(K_3)} (X_{u',v'} - X_{v',u'}) \cdot (u', v') \right) \mod 0_{G(K_3)} \]
implies \( X_{u',v'} \equiv X_{v',u'} \mod A(K_3) \) for all \( (u', v') \in A(K_3) \). We then have 
\[ \sum_{(u', v') \in A(H')} X_{u',v'} \]
even, which is incompatible with the parity constraint (2). Therefore the system of 
equations (1), (2), (3) of Proposition 6 has no integer solutions on \( K_3 \). It follows that 
if this system has integer solutions on a graph \( H \), then \( H \) admits no homomorphism 
to \( K_3 \), whence \( \chi(H) \geq 4 \).
For a graph $H$, the system of equations of Proposition 6 can be resolved efficiently. If there are no solutions, this allows to conclude that $\text{coind}(B((H))) + 2 \leq 3$. If there are solutions, this allows to conclude that $\chi(H) \geq 4$. Though the converse of Proposition 5 does not hold, it is not known whether the existence of solutions to the system of equations of Proposition 6 implies that $\text{coind}(B((H))) + 2 \geq 4$. In the affirmative, this would imply that deciding whether an input graph $H$ satisfies $\text{coind}(B((H))) + 2 \leq 3$ can be done in polynomial time. Otherwise the system of equations of Proposition 6 provides an effective computation that allows to determine non 3-colourability on a class of graphs that strictly includes the class of graphs $H$ that satisfy $\text{coind}(B(H)) + 2 \geq 4$. In any case it would be interesting to know whether similar methods can be devised for larger chromatic numbers.

In the next section we give an example showing what goes wrong with the converse of Proposition 5, and propose reasonable alternatives. In the following sections, we give practical examples of computations of the system of Proposition 6.

4. Example: the fallacy of the converse of Proposition 5

Consider the graph $H$ in Figure 1. Let $f : C_3 \to H$ be the homomorphism defined by $f(0) = A$, $f(1) = C$ and $f(2) = B$. Any neighbour $g$ of $f$ is in the set $S$ defined by

$$S = \{ g \in H^{K_3} : g(0) \in \{A, 0''\}, g(1) \in \{C, 3''\}, g(2) \in \{B, 6''\} \}.$$  

It is easy to check that any neighbour of an element of $S$ is again in $S$. Thus $f$ is not in the connected component of the constants in $H^{C_3}$. We will show that $\sigma_{L(H)}(f) = 1_{\sigma_{L(H)}}$.

Let $C_9$ be the 9-cycle with vertex-set $\mathbb{Z}_9$. Let $h_0, h_1, h_2, h_3, h_4 : \mathbb{Z}_9 \to V(H)$ be defined as follows: $h_0(i) = U$ for all $i \in \mathbb{Z}_9$, $h_1, h_2, h_3$ are defined by $h_1(i) = i$, $h_2(i) = i'$, $h_3(i) = i''$, and $h_4$ is defined by

$$(h_4(0), h_4(1), \ldots, h_4(8)) = (A, C, A, C, B, C, B, A, B).$$

![Figure 1. $H$](image)
Since \( h_0 \) is a constant and \( h_0, h_1, h_2, h_3, h_4 \) is a path in \( H^{C_0} \), we have \( \sigma_{G^*}(H)(h_4) = 1_{G^*}(H) \). Note that \( h_4 \) is a homomorphism, hence \( \sigma_{G^*}(H)(h_4) = \sigma_{F^*}(A(H))(h_4, h_4) / \theta^* \).

By definition we have
\[
\sigma_{F^*}(A(H))(h_4, h_4) = (A, C) \cdot (A, C)^{-1} \cdot (A, C) \cdot (B, C)^{-1} \cdot (B, C) \cdot (B, C)^{-1} \\
\cdot (B, A) \cdot (B, A)^{-1} \cdot (B, A) \cdot (C, A)^{-1} \cdot (C, A) \cdot (C, A)^{-1} \\
\cdot (C, B) \cdot (C, B)^{-1} \cdot (C, B) \cdot (A, B)^{-1} \cdot (A, B) \cdot (A, B)^{-1} \\
= (A, C) \cdot (B, C)^{-1} \cdot (B, A) \cdot (C, A)^{-1} \cdot (C, B) \cdot (A, B)^{-1} \\
= \sigma_{F^*}(A(H))(f, f).
\]

Therefore \( \sigma_{G^*}(H)(f) = \sigma_{F^*}(A(H))(f, f) / \theta^* = 1_{G^*}(H) \).

Note that \( h_4 : C_9 \to H \) factors as \( f \circ f' \), with \( f' : C_9 \to C_3 \) given by \( (f'(0), f'(1), \ldots, f'(8)) = (0, 1, 0, 1, 2, 1, 2, 0, 2) \).

Therefore \( h_4 \) could be seen as an “unfolding” of \( f \), sufficient to fall in the connected component of the constants. It is not clear whether a similar phenomenon always occurs.

**Problem 9.** Let \( H \) be a graph, \( C_n \) an odd cycle and \( f \) a homomorphism in \( H^{C_n} \) such that \( \sigma_{G^*}(H)(f) = 1_{G^*}(H) \). Does there exist an odd cycle \( C_m \) and a homomorphism \( f' : C_m \to C_n \) such that \( f \circ f' \) is in the connected component of a constant in \( H^{C_m} \)?

**Problem 10.** Let \( H \) be a graph, \( C_n \) an odd cycle and \( f \) a homomorphism in \( H^{C_n} \) such that \( \sigma_{G^*}(H)(f) = 0_{G^*}(H) \). Does there exist an odd cycle \( C_m \) and a homomorphism \( f' : C_m \to C_n \) such that \( f \circ f' \) is in the connected component of a constant in \( H^{C_m} \)?

5. **Example: \( K_4 \)**

In the previous section, we avoided computing \( \mathcal{G}(H) \). In this section we consider basic examples. We noted in the proof of Proposition 8 that if \( H \) has no 4-cycles, then \( \mathcal{G}(H) = \mathcal{F}(A(H)) \simeq \mathbb{Z}^{A(H)} \) and the linear system of Proposition 6 has no solutions. Of course it is well known that if \( H \) is square-free, then \( \text{coind}(B(H)) + 2 \leq 3 \).

![Figure 2. \( K_4 \)](image)

Now consider \( H = K_4 \) with vertex-set \( \{0, 1, 2, 3\} \). We determine the structure of \( \mathcal{G}(K_4) \) by finding a basis within \( A(K_4) \). Consider the arcs \( a = (0, 1) \), \( b = (1, 2) \) and \( c = (2, 3) \). For \( e = (i, j) \), we will write \( e^{-\alpha} \) for \(-\alpha(e) = (j, i) \). By definition of \( \theta \) we have
\[
(0, 3) / \theta = (a - b^{-\alpha} + c) / \theta, \\
(3, 0) / \theta = (c^{-\alpha} - b + a^{-\alpha}) / \theta.
\]
We need to add an arc, say \( d = (0, 2) \) to express the remaining arcs modulo \( \theta \):

\[
\begin{align*}
(1,3)/\theta &= (b-d+(0,3))/\theta = (b-d+a-b^{-\alpha}+c)/\theta, \\
(3,1)/\theta &= (c^{-\alpha}-d+a)/\theta, \\
(2,0)/\theta &= (c-(1,3)+a^{-\alpha})/\theta = (b^{-\alpha}-a+d-b+a^{-\alpha})/\theta.
\end{align*}
\]

It is easy to check that every sum \((x,y) - (z,y) + (z,w) - (x,w))/\theta\) expressed in terms of \( B = \{a,a^{-\alpha},b,b^{-\alpha},c,c^{-\alpha},d\} \) is 0. Therefore \( \mathcal{G}(K_4) \simeq \mathbb{Z}^B \). Note that \( B \) corresponds to a spanning tree in \( K_4 \times \bar{K}_2 \), where \( \bar{K}_2 \) is the transitive tournament with one arc. Such spanning trees of \( H \times \bar{K}_2 \) always correspond to copies of \( \mathbb{Z}^{2|V(H)|} \) in \( \mathcal{G}(H) \). In general we have the following.

**Remark 11.** For any graph \( H \), there exists a set \( B \in A(H) \) such that \( \mathcal{G}(H) \simeq \mathbb{Z}^B \).

**Proof.** Let \( S \) be a set of 4-cycles in \( H \times \bar{K}_2 \) such that the corresponding equations

\[
(u,v) - (w,v) + (w,x) - (u,x) \equiv 0_{\mathcal{F}(A(H))}
\]

form a basis of the kernel of \( \theta \). Consider the bipartite graph whose parts are \( S \) and \( A(H) \), with an element of \( S \) being linked to the four arcs it contains. Since \( S \) is a basis, any subset \( S' \) of \( S \) has at least \( |S'| \) neighbours in \( A(H) \). Therefore this bipartite graph has a matching that saturates \( S \). For \( s \in S \), let \( m(s) \) be the arc of \( H \) matched to \( s \).

Put \( S = \{s_1, \ldots, s_k\} \). For \( i = 1, \ldots, k \). We can recursively express \( m(s_i)/\theta \) in terms of \( \theta \)-classes of the arcs in \( A(G) \setminus \{m(s_1), \ldots, m(s_i)\} \) in the equations corresponding to \( s_j, j \geq i \): If the equation corresponding to \( s_i \) is

\[
(u,v) - (w,v) + (w,x) - (u,x) \equiv 0_{\mathcal{F}(A(H))}
\]

and \( m(s_i) = (u,v) \), we put

\[
(u,v)/\theta = ((w,v) - (w,x) + (u,x))/\theta.
\]

(Note that some of \( (w,v)/\theta, (w,x)/\theta, (u,x)/\theta \) may already have appeared as \( m(s_j), j < i \), and been replaced accordingly.) Since \( S \) is a basis of the kernel of \( \theta \), all the constraints corresponding to the remaining cycles will be satisfied. Therefore \( \mathcal{G}(H) \simeq \mathbb{Z}^B \), where \( B = A(H) \setminus m(S) \).

We now return to the case study of \( K_4 \). Note that every homomorphism \( f : C \to K_4 \) is at distance at most two from any constant in \( K_4^C \), therefore we must have \( \sigma_{\mathcal{G}(K_4)}(f) = 0_{\mathcal{G}(K_4)} \). In terms of the signature constraint 3 of Proposition 6, this gives one equation in \( X_{u,v}, (u,v) \in A(K_4) \) for each element of the basis \( B \) of \( \mathcal{G}(K_4) \):

\[
\begin{align*}
a: (X_{0,1} - X_{1,0}) + (X_{0,3} - X_{3,0}) + (X_{1,3} - X_{3,1}) + (X_{3,1} - X_{1,3}) - (X_{2,0} - X_{0,2}) &= 0, \\
a^{-\alpha}: (X_{1,0} - X_{0,1}) + (X_{3,0} - X_{0,3}) + (X_{2,0} - X_{0,2}) &= 0, \\
b: (X_{1,2} - X_{2,1}) - (X_{3,0} - X_{0,3}) + (X_{1,3} - X_{3,1}) - (X_{2,0} - X_{0,2}) &= 0, \\
b^{-\alpha}: (X_{2,1} - X_{1,2}) - (X_{0,3} - X_{3,0}) - (X_{1,3} - X_{3,1}) + (X_{2,0} - X_{0,2}) &= 0, \\
c: (X_{2,3} - X_{3,2}) + (X_{0,3} - X_{3,0}) + (X_{1,3} - X_{3,1}) &= 0, \\
c^{-\alpha}: (X_{3,2} - X_{2,3}) + (X_{3,0} - X_{0,3}) + (X_{1,3} - X_{3,1}) &= 0, \\
d: (X_{0,2} - X_{2,0}) - (X_{1,3} - X_{3,1}) - (X_{3,1} - X_{1,3}) + (X_{2,0} - X_{0,2}) &= 0.
\end{align*}
\]
The equation corresponding to \( d \) is trivially satisfied. The equations corresponding to \( c \) and \( c^{-\alpha} \) are instances of the flow constraints (1), and with a bit of tampering, we see that the equations corresponding to \( a, a^{-\alpha}, b \) and \( b^{-\alpha} \) are also consequences of the flow constraints. Therefore the signature constraint is redundant, as expected.

6. Example: \( K_{7/2} \)

For integers \( s, r \) such that \( 1 \leq r \leq s/2 \), the circular graph \( K_{s/r} \) is the graph with \( V(K_{s/r}) = \mathbb{Z}_s \) and

\[
E(K_{s/r}) = \{[i, j] : j - i \in \{r, r + 1, \ldots, s - r\}\}.
\]

It is well known that \( \chi(K_{s/r}) = \lceil s/r \rceil \) and \( \text{coind}(B(K_{s/r})) + 2 = 3 \) if \( 2 < s/r < 4 \).

We will examine the case of \( K_{7/2} \). We adopt a non standard presentation: we put \( V(K_{7/2}) = \mathbb{Z}_7 \) and

\[
E(K_{7/2}) = \{[i, i+1] : i \in \mathbb{Z}_7\} \cup \{[i, i+2] : i \in \mathbb{Z}_7\}
\]

as in the following figure. (The standard presentation is obtained by multiplying every vertex label by 2.)

\[\text{Figure 3. } K_{7/2}\]

Put \( a = (4,5), b = (5,6), c = (6,0), d = (6,1), e = (0,1), f = (1,2) \) and \( g = (2,3) \). The set \( B = \{a, \ldots, g\} \cup \{a^{-\alpha}, \ldots, g^{-\alpha}\} \) can be used to express the
remaining arcs of $K_{7/2}$: We have
\[
\begin{align*}
(5,0)/\theta &= (b - d^{-\alpha} + e^{-\alpha})/\theta, \\
(0,5)/\theta &= (e - d + b^{-\alpha})/\theta, \\
(0,2)/\theta &= (e^{-\alpha} - d^{-\alpha} + f)/\theta, \\
(2,0)/\theta &= (f^{-\alpha} - d + c)/\theta, \\
(4,6)/\theta &= ((4,5) - (0,5) + (0,6))/\theta \\
&= (a - e + d - b^{-\alpha} + e^{-\alpha})/\theta, \\
(6,4)/\theta &= (c - b + d^{-\alpha} - e^{-\alpha} + a^{-\alpha})/\theta, \\
(3,1)/\theta &= ((3,2) - (0,2) + (0,1))/\theta \\
&= (g^{-\alpha} - c^{-\alpha} + d^{-\alpha} - f + e)/\theta, \\
(1,3)/\theta &= (e^{-\alpha} - f^{-\alpha} + d - c + g)/\theta, \\
(2,4)/\theta &= ((2,1) - (6,1) + (2,4))/\theta \\
&= (f^{-\alpha} - d + c - b + d^{-\alpha} - e^{-\alpha} + a^{-\alpha})/\theta, \\
(4,2)/\theta &= (a - e + d - b^{-\alpha} + c^{-\alpha} - d^{-\alpha} + f)/\theta, \\
(3,5)/\theta &= ((3,1) - (6,1) + (6,5))/\theta \\
&= (g^{-\alpha} - c^{-\alpha} + d^{-\alpha} - f + e - d + b^{-\alpha})/\theta, \\
(5,3)/\theta &= (b - d^{-\alpha} + e^{-\alpha} - f^{-\alpha} + d - c + g)/\theta, \\
(3,4)/\theta &= ((3,1) - (6,1) + (6,4))/\theta \\
&= (g^{-\alpha} - c^{-\alpha} + d^{-\alpha} - f + e - d + c - b + d^{-\alpha} - e^{-\alpha} + a^{-\alpha})/\theta, \\
(4,3)/\theta &= (a - e + d - b^{-\alpha} + c^{-\alpha} - d^{-\alpha} + e^{-\alpha} - f^{-\alpha} + d - c + g)/\theta.
\end{align*}
\]
Note that $-\alpha(B) = B$ so that the expression for $(v, u)$ is obtained by applying $-\alpha$ to $(u, v)$ and its expression. This was not the case for $K_4$, which had an asymmetric basis.

The 4-cycles of $K_{7/2}$ have the form $i, i + 1, i + 3, i + 2$ or $i, i + 2, i + 4, i + 6$. It is easy to verify that the corresponding congruences, expressed in terms of $B$, are all satisfied. Therefore $G(H) \simeq \mathbb{Z}_B$.

We will now show that the linear system of Proposition 6 is inconsistent. We work in terms of the variables $Y_{i,j} = X_{i,j} - X_{j,i}$. We have $Y_{j,i} = -Y_{i,j}$, therefore we can express the system in terms of the variables $Y_{i,j}$ with

\[(i, j) \in \mathcal{O} = \{(i, i + 1) : i \in \mathbb{Z}_7\} \cup \{(i, i + 2) : i \in \mathbb{Z}_7\}.
\]

The flow constraints (1) can be expressed as

\[Y_{i-2,i} + Y_{i-1,i} = Y_{i,i+1} + Y_{i,i+2}\]

for every $i \in \mathbb{Z}_7$. Also, we have

\[\sum_{(i,j) \in \mathcal{A}(H)} X_{i,j} = \sum_{(i,j) \in \mathcal{O}} Y_{i,j} + 2 \cdot \sum_{(i,j) \not\in \mathcal{O}} X_{i,j}\]

hence with $M = N - \sum_{(i,j) \not\in \mathcal{O}} X_{i,j}$, the parity constraint (2) is

\[\sum_{(i,j) \in \mathcal{O}} Y_{i,j} - 2M = 1.\]

The signature constraint (3) gives one equation for each element of $B$. It can be verified that many of these are consequences of the flow constraints (1), as was the
case for $K_4$. Consider the equation corresponding to $c$:

$$Y_{6,0} - Y_{0,2} - Y_{4,6} - Y_{1,3} + Y_{2,4} + Y_{3,5} + Y_{3,4} + Y_{4,5} = 0.$$ 

The flow constraints allow to replace $-Y_{1,3} + Y_{3,4} + Y_{3,5}$ by $Y_{2,3}$ and $-Y_{4,6} + Y_{2,4} + Y_{3,4}$ by $Y_{4,5}$. With these substitutions, we get

$$Y_{6,0} - Y_{0,2} + Y_{2,3} + Y_{4,5} = 0,$$

that is, $F(6) = 0$, with

$$F(i) = Y_{i,i+1} - Y_{i+1,i+3} + Y_{i+3,i+4} + Y_{i+5,i+6}$$

for all $i \in \mathbb{Z}_7$. The flow constraint at $i + 1$ allows to replace $Y_{i,i+1} - Y_{i+1,i+3}$ by $Y_{i+1,i+2} - Y_{i-1,i+1}$. We get

$$F(i) = Y_{i+5,i+6} - Y_{i-1,i+1} + Y_{i+1,i+2} + Y_{i+3,i+4} = F(i + 5).$$

Hence $F(i) = 0$ for all $i \in \mathbb{Z}_7$. Therefore

$$\sum_{i \in \mathbb{Z}_7} F(i) = 3 \cdot \sum_{i \in \mathbb{Z}_7} Y_{i,i+1} - \sum_{i \in \mathbb{Z}_7} Y_{i,i+2} = 0.$$

Adding this to the parity constraint gives

$$4 \cdot \sum_{i \in \mathbb{Z}_7} Y_{i,i+1} - 2M = 1,$$

which has no integer solutions. This proves that $\text{coind}(B(K_7/2)) + 2 \leq 3$.

References


