NEAR-UNANIMITY POLYMORPHISMS ON STRUCTURES
WITH FINITE DUALITY

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Abstract. We introduce a combinatorial parameter on finite relational trees, called the degree of monstrosity, which measures the smallest possible arity of a near-unanimity polymorphism on a core structure with finite duality. We also show that the core structures which admit all conservative operations as polymorphisms are essentially the structures with finite duality whose minimal obstructions are trees with degree of monstrosity at most one.

1. Introduction

An operation \( f : X^n \rightarrow X \) is called a near-unanimity operation if it satisfies the near-unanimity identities
\[
f(y, x, x, \ldots, x) \approx f(x, y, x, \ldots, x) \approx \cdots \approx f(x, x, x, \ldots, y) \approx x.
\]
In [6], Maróti solved a long-standing problem by providing a procedure which decides whether a finite set of operations on a finite universe generates an algebra with a near-unanimity operation. In contrast, no procedure is known to decide whether a given finite relational structure admits a near-unanimity polymorphism. The latter problem is relevant from the point of view of descriptive complexity: In [3], it is shown that the existence of a near-unanimity polymorphism on a relational structure \( A \) implies that the constraint satisfaction problem for \( A \) is polynomial-time solvable, and in particular it has “bounded width” in the sense of [2]. There is no known procedure to decide whether a constraint satisfaction problem has bounded width. However, at the lower end of the bounded width hierarchy, we find the concept of “tree duality” (or “width one”) for which a decision procedure is known [2]. The simplest subclass of structures with tree duality is the class of structures with “finite duality” (or “first-order definable constraint satisfaction problem”), which has been characterised in [5, 7]. In [5] it is shown that every core structure with finite duality admits a near-unanimity polymorphism, though not every core structure with tree duality admits a near-unanimity polymorphism. Perhaps a better understanding of near-unanimity polymorphisms on core structures with tree duality would trigger progress on the near-unanimity problem for general relational structures.

The near-unanimity operations of arity three are called majority operations. In [4], we gave a characterisation of the core structures with finite duality which admit majority polymorphisms, in terms of a combinatorial description of their minimal

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obstructions as “caterpillars”. This “caterpillar duality” was formalised through datalog by Carvalho, Dalmau and Krokhin [1]. The property is decidable and characterises the structures which are homomorphically equivalent to a structure with lattice polymorphisms. In the present paper, we describe a combinatorial parameter on relational trees — the “degree of monstrosity” — which we use in Corollary 4.4 to characterise the minimal arity of a near unanimity polymorphism on a core structure with finite duality.

2. Preliminaries

Basics. A vocabulary is a finite set $\sigma = \{R_1, \ldots, R_m\}$ of relation symbols, each with an arity $r_i$ assigned to it. A $\sigma$-structure is a relational structure $A = \langle A; R_1(A), \ldots, R_m(A) \rangle$ where $A$ is a non-empty set called the universe of $A$, and $R_i(A)$ is an $r_i$-ary relation on $A$ for each $i$. The elements of $R_i(A)$, $1 \leq i \leq m$ will be called hyperedges of $A$. We will usually use the same letter in boldface and slanted type to denote a relational structure and its universe, when no confusion is possible.

Homomorphisms. For $\sigma$-structures $A$ and $B$, a homomorphism from $A$ to $B$ is a map $f : A \rightarrow B$ such that $f(R_i(A)) \subseteq R_i(B)$ for all $1 \leq i \leq m$, where for any relation $R \in \sigma$ of arity $r$ we have

$$f(R) = \{(f(x_1), \ldots, f(x_r)) : (x_1, \ldots, x_r) \in R\}.$$ 

We write $A \rightarrow B$ if there exists a homomorphism from $A$ to $B$, and $A \not\rightarrow B$ otherwise. We write $A \twoheadrightarrow B$ when $A \rightarrow B$ and $B \rightarrow A$: $A$ and $B$ are then called homomorphically equivalent. For a finite structure $A$, we can always find a structure $B$ such that $A \twoheadrightarrow B$ and the cardinality of the universe of $B$ is minimal with respect to this property. It is well known (see [7]) that any two such structures are isomorphic. We then call $B$ the core of $A$. A structure $A$ is called rigid if the identity is the only homomorphism from $A$ to itself.

Products. We define the product of two $\sigma$-structures $A$ and $B$ as the structure

$$A \times B = \langle A \times B; R_1(A \times B), \ldots, R_m(A \times B) \rangle,$$

where, for $i = 1, \ldots, m$

$$R_i(A \times B) = \{((a_1, b_1), \ldots, (a_r, b_r)) : (a_1, \ldots, a_r) \in R_i(A), (b_1, \ldots, b_r) \in R_i(B)\}.$$ 

The product is associative and commutative. We write $\prod_{i \in I} A_i$ for the product of a family $\{A_i : i \in I\}$ of $\sigma$-structures. When all factors are equal, we use the power notation and write $A^n$ for $\prod_{i=1}^n A$. A polymorphism on $A$ is a homomorphism from $A^n$ to $A$ for some $n$.

Trees. Let $A$ be a $\sigma$-structure. We define the incidence multigraph $\text{Inc}(A)$ of $A$ as the bipartite multigraph with parts $A$ and

$$\text{Block}(A) = \{(R, (x_1, \ldots, x_r)) : R \in \sigma \text{ has arity } r \text{ and } (x_1, \ldots, x_r) \in R(A)\},$$

and with edges $e_{a,i,B}$ joining $a \in A$ to $B = (R, (x_1, \ldots, x_r)) \in \text{Block}(A)$ when $x_i = a$. Thus, $\text{Inc}(A)$ is irreflexive but may have 2-cycles, that is, parallel edges. A $\sigma$-structure $T$ is called a $\sigma$-tree (or tree for short) if $\text{Inc}(T)$ is a (graph-theoretic) tree, that is, it is connected and has no cycles or parallel edges. An element of a $\sigma$-tree is called a leaf if it is incident to only one block, and a non-leaf otherwise. Similarly, a block of a $\sigma$-tree is called pendant if it is incident to at most one non-leaf element, and non-pendant otherwise.
Finite duality. We will use the following results on finite duality.

**Theorem 2.1 ([7]).** For every \( \sigma \)-tree \( T \), there exists a \( \sigma \)-structure \( D(T) \) such that for every \( \sigma \)-structure \( B \), we have

\[
B \rightarrow D(T) \iff T \not\rightarrow B.
\]

We give in Section 4 the construction given in [8] for \( D(T) \) (see Theorem 4.1). A structure \( D \) is said to have finite duality if there is a finite family \( F = \{A_1, \ldots, A_m\} \) of core structures such that for every \( \sigma \)-structure \( B \), we have

\[
B \rightarrow D \iff \text{there exists } A_i \in F \text{ such that } A_i \not\rightarrow B.
\]

**Theorem 2.2 ([7]).** A structure \( D \) has finite duality if and only if there exists a family \( F = \{T_1, \ldots, T_m\} \) of trees such that \( D \leftarrow \prod_{i=1}^m D(T_i) \).

By retracting to cores and removing redundant structures, we can assume that \( F \) contains only essential core elements, that is, \( T_i, T_j \in F \) with \( i \neq j \) implies \( T_i, T_j \) are core trees and \( T_i \not\rightarrow T_j \). The family \( F \) is then uniquely determined by \( D \), and the elements of \( F \) are called the minimal obstructions of \( D \).

### 3. Untangled Sets and Degrees of Monstrosity

Let \( T \) be a \( \sigma \)-tree. A set \( S \) of non-leaves of \( T \) is called untangled if there is no triple of distinct elements \( x, y, z \in S \) such that \( x \) is on the path from \( y \) to \( z \) (in \( \text{Inc}(T) \)).

**Lemma 3.1.** Let \( A \) be a structure with finite duality, and \( k \geq 2 \), an integer. If \( A \) has a minimal obstruction \( T \) which contains an untangled set \( S \) such that \( |S| = k \), then \( A \) does not admit a near-unanimity polymorphism of arity \( k \).

**Proof.** Let \( S = \{u_1, \ldots, u_k\} \). We define the subtrees \( T_0, T_1, \ldots, T_k \) of \( T \) as follows: for \( i = 1, \ldots, k \), \( T_i \) is the subtree spanned by all the blocks \( B \) such that \( u_i \) lies on the path (in \( \text{Inc}(T) \)) from \( B \) to any other element \( u_j \) of \( S \); \( T_0 \) is the subtree spanned by the remaining blocks of \( T \). By the definition of \( S \), the trees \( T_1, \ldots, T_k \) each contain at least one block, and every element \( u_i \) of \( S \) is pendant in \( T_0 \).

Let \( C_1, \ldots, C_k \) be \( k \) copies of \( T_0 \), where an element \( u \) of \( T_0 \) is labeled \((u, i)\) in \( C_i \). We build the structure \( B \) from the disjoint union of \( T_1, \ldots, T_k, C_1, \ldots, C_k \) with the following identifications: For \( i = 1, \ldots, k \) we identify the element \( u_i \) in \( T_i \) to all elements \((u_i, j)\) in \( C_j \) for all \( j \neq i \). Note that \( B \) admits a natural homomorphism \( f : B \rightarrow T \), where the elements of \( S \) have exactly two preimages, the other elements of \( T_1, \ldots, T_k \) have exactly one preimage and the other elements of \( T_0 \) have exactly \( k \) preimages.

We now show that there is no homomorphism from \( T \) to \( B \). Suppose that \( g : T \rightarrow B \) is a homomorphism. Since \( T \) is a core, \( f \circ g \) is an automorphism. By [4], Lemma 2.4, \( T \) is rigid, hence \( f \circ g \) is the identity on \( T \). Therefore \( g \) must map every element \( u \) of \( T \) to an element of \( f^{-1}(u) \). In particular, the non-leaf \( u_1 \in S \) cannot be mapped to the leaf \((u_1, 1)\) of \( C_1 \), so \( g(u_1) \) must be the copy of \( u_1 \) in \( T_1 \). Let \( B \) be the only block of \( T_0 \) containing \( u_1 \). Then \( g \) maps \( B \) to its copy in \( C_j \) for some \( j \neq i \). But then, by connectivity \( g \) must map \( u_j \) to \((u_j, j)\), which is impossible since \( u_j \) is not a leaf in \( T \). Thus, \( T \not\rightarrow B \). And since \( B \rightarrow T \), it means that no minimal obstruction of \( A \) admits a homomorphism to \( B \). Therefore there exists a homomorphism \( h : B \rightarrow A \).
Now suppose that \( A \) admits a near-unanimity polymorphism \( \nu : A^k \rightarrow A \). Then we can define a homomorphism \( \phi_0 : T_0 \rightarrow A \) by
\[
\phi_0(u) = \nu(h(u, 1), \ldots, h(u, k)).
\]
By the near-unanimity identity, we have \( \phi_0(u_i) = h(u_i) \) for \( i = 1, \ldots, k \). Thus we can combine \( \phi_0 \) with the homomorphisms \( \phi_i : T_i \rightarrow A, i = 1, \ldots, k \) where \( \phi_i(u) = h(u) \), to get a homomorphism \( \phi : T \rightarrow A \), which is impossible. Thus \( A \) does not admit a near-unanimity polymorphism.

An untangled set \( S \) of \( T \) is called maximally stretched if each element of \( S \) is incident to at most one non-pendant block. Every untangled set can be turned into a maximally stretched untangled set by pulling on it. \( T \) contains a unique maximally stretched untangled set of maximum cardinality \( S_T \), constructed from \( \text{Inc}(T) \) as follows: We first remove the leaves of the part \( T \), then the leaves (pendant blocks) of \( \text{Block}(T) \). The leaves in what remains of \( \text{Inc}(T) \) are all in \( T \); they constitute the set \( S_T \). We call the cardinality of \( S_T \) the degree of monstrosity \( \mu(T) \) of \( T \).

(In various mythologies, the “monstrosity” of a creature is indeed measured by counting its limbs after it is shaved off.)

The caterpillars of \([4]\) are the trees \( T \) such that \( \mu(T) \leq 2 \), and by \([4]\), Corollary 4.3, these are the minimal obstructions of the core structures with finite duality which admit a near-unanimity polymorphism of arity 3. In the next section, we will show that for every integer \( k \geq 2 \), the trees with degree of monstrosity at most \( k \) are the minimal obstructions of the core structures with finite duality which admit a near-unanimity polymorphism of arity \( k + 1 \). A tree \( T \) such that \( \mu(T) \leq 1 \) is called a medusa. (Note that \( \mu(T) = 0 \) if and only if \( T \) has only one hyperedge.) In Section 5, we investigate the algebraic meaning of medusa duality.

4. Duals of relational trees and near-unanimity polymorphisms

Let \( \sigma \) be a finite vocabulary and \( T \) a \( \sigma \)-tree. Following \([8]\), we define its dual \( D(T) \) as a structure on the universe
\[
D(T) = \{ f : T \rightarrow \text{Block}(T) : [a, f(a)] \in E(\text{Inc}(T)) \text{ for all } a \in T \}
\]
as follows. For \( R \in \sigma \) of arity \( r \), we let \( R(D(T)) \) be the set of all \( r \)-tuples \( (f_1, \ldots, f_r) \in D(T)^r \) such that for all \( (x_1, \ldots, x_r) \in R(T) \), there exists \( j \in \{1, \ldots, r\} \) such that \( f_j(x_j) \neq (R, (x_1, \ldots, x_r)) \). The fundamental property of this construction is the following.

**Theorem 4.1** \([8]\). Let \( A \) be a \( \sigma \)-structure. Then \( A \rightarrow D(T) \) if and only if \( T \not\rightarrow A \).

In general, \( D(T) \) is not a core. However it contains a copy \( D \) of its core as an induced substructure. In particular, the universe \( D \) of \( D \) consists of functions \( f : T \rightarrow \text{Block}(T) \).

**Lemma 4.2.** For every core tree \( T \) the core \( D \) of \( D(T) \) admits a near-unanimity polymorphism of arity \( \max\{3, \mu(T) + 1\} \).

*Proof.* If \( \mu(T) \leq 2 \), then \( T \) is a caterpillar and \( D \) admits a majority polymorphism by \([4]\). Thus we may assume \( \mu(T) \geq 3 \). (Though the proof given below is valid for \( \mu(T) = 2 \).)
Let \( \sim \) be the equivalence on \( D^{\mu(T)+1} \) which identifies \((f_0, \ldots, f_\mu(T)) \) to \((f, \ldots, f)\) if there are at least \( \mu(T) \) indices \( i \) such that \( f_i = f \). We show that there is no homomorphism from \( T \) to the quotient \( D^{\mu(T)+1}/\sim \).

Suppose that there exists a homomorphism \( \phi : T \rightarrow D^{\mu(T)+1}/\sim \). For an element \( u \) of \( T \), \( \phi(u) \) is an equivalence class which is either a singleton of the form \((f_0, \ldots, f_\mu(T), u)\), or the set of all \((\mu(T) + 1)\)-tuples differing from \((f_0, \ldots, f_u)\) in at most one coordinate. We define the \((u, \phi)\)-weight \( w_{u, \phi}(B) \) of a block \( B \) incident to \( u \) as follows:

\[
w_{u, \phi}(B) = \begin{cases} \{i : (\mu(T) + 1)\} & \text{if } \phi(u) \text{ is the singleton } (f_0, \ldots, f_\mu(T), u). \\ 0 & \text{if } \phi(u)/\sim = (f_0, \ldots, f_u)/\sim \text{ where } f_u(u) = B. \\ \end{cases}
\]

Let \( u \) be a non-leaf element of \( T \) and \( P = (R, (x_1, \ldots, x_\mu)) \) a pendant block incident to \( u \) with \( u = x_k \), then \( w_{u, \phi}(P) = 0 \). Indeed, since \( \phi \) preserves \( P \), for \( j = 1, \ldots, r \), there exists \((f_0, x_j, \ldots, f_\mu(T), x_j) \in \phi(x_j) \) such that

\[
(f_0, x_1, \ldots, f_\mu(T), x_1), \ldots, (f_0, x_r, \ldots, f_\mu(T), x_r) \in R(D^{\mu(T)+1}).
\]

By definition of \( R(D) \), for all \( i = 0, \ldots, \mu(T) \), there exists \( j(i) \) such that \( f_{i,j(i)}(x_{j(i)}) \neq P \). However since \( P \) is pendant the only possibility is to have \( j(i) = k \) for all \( i \). Thus \( \phi(x_k)/\sim = \phi(u)/\sim \) contains a \((\mu(T) + 1)\)-tuple with no coordinates mapping \( u \) to \( P \), whence \( w_{u, \phi}(P) = 0 \).

Now let \( S = \{s_1, \ldots, s_\mu(T)\} \) be a maximally stretched untangled set in \( T \). For an element \( u \) of \( T \) and a block \( B \) incident to \( u \), we define the \((u, S)\)-weight \( w_{u, S}(B) \) of \( B \) and the \((B, S)\)-weight \( w_{B, S}(u) \) of \( u \) as follows:

\[
w_{u, S}(B) &= |\{i : B \text{ is on the path from } u \text{ to } s_i \text{ in Inc}(T)\}|, \\
w_{B, S}(u) &= |\{i : u \text{ is on the path from } B \text{ to } s_i \text{ in Inc}(T)\}|.
\]

Note that if \( u \) is incident to \( B \), then \( w_{u, S}(B) + w_{B, S}(u) = \mu(T) \). We have \( w_{B, S}(u) = 0 \) if and only if \( u \) is a leaf, and if \( u \) is a non-leaf, \( w_{u, S}(B) = 0 \) if and only if \( B \) is pendant. The sum of \((B, S)\)-weights of elements incident to \( B \) is always \( \mu(T) \), and the sum of \((u, S)\)-weights of blocks incident to \( u \) is \( \mu(T) - 1 \) if \( u \in S \), and \( \mu(T) \) otherwise. In particular, since the set of blocks incident to any element \( u \) has total \((u, \phi)\)-weight \( \mu(T) + 1 \), there exists a block \( B_u \) incident to \( u \) such that \( w_{u, S}(B_u) < w_{u, \phi}(B_u) \).

**Fact.** Every block \( B \) admits an incident element \( u_B \) such that \( w_{B, S}(u_B) + w_{u_B, \phi}(B) \leq \mu(T) \).

**Proof of Fact.** Let \( B = (R, (x_1, \ldots, x_r)) \). Since \( \phi \) preserves \( B \), for \( j = 1, \ldots, r \), there exists \((f_0, x_j, \ldots, f_\mu(T), x_j) \in \phi(x_j) \) such that

\[
(f_0, x_1, \ldots, f_\mu(T), x_1), \ldots, (f_0, x_r, \ldots, f_\mu(T), x_r) \in R(D^{\mu(T)+1}).
\]

By definition of \( R(D) \), for all \( i = 0, \ldots, \mu(T) \), there exists \( j(i) \) such that \( f_{i,j(i)}(x_{j(i)}) \neq B \). However, since \( \sum_{j=1}^r w_{B, S}(x_j) < \mu(T) + 1 \), there exists an index \( j \) such that \( |\{i : j(i) = j\}| > w_{B, S}(x_j) \). Note that \( x_j \) cannot be a leaf, thus \( w_{B, S}(x_j) \geq 1 \), whence \( |\{i : j(i) = j\}| \geq 2 \). If \( \phi(x_j)/\sim \) is a singleton, then \( \mu(T) + 1 - w_{x_j, \phi}(B) \geq |\{i : j(i) = j\}| \); otherwise \( w_{x_j, \phi}(B_u) \geq 0 \) since \( \phi(x_j)/\sim \) contains a \((\mu(T) + 1)\)-tuple with at least two coordinates mapping \( x_j \) away from \( B \). In any case, for \( u_B = x_j \), we have \( w_{B, S}(u_B) + w_{u_B, \phi}(B) \leq \mu(T) \).
Therefore we can define an infinite walk $B_0, u_1, B_1, u_2, B_2, \ldots$ in $\text{Inc}(T)$ recursively. We select the block $B_0$ arbitrarily, and when $B_j$ is already chosen, we select $u_{i+1}$ incident to $B_i$ such that $w_{B_i}, s(u_{i+1}) + w_{u_{i+1}}, \phi(B_i) \leq \mu(T)$. Since 
\[
\sum \{w_{u_{i+1}}, \phi(B) : B \neq B_i \text{ is incident to } u\} = \mu(T) + 1 - w_{u_{i+1}}, \phi(B_i) 
\geq w_{B_i}, s(u_{i+1}) + 1 
\leq \mu(T) - w_{u_{i+1}}, s(B_i) + 1,
\]
there exists a block $B_{i+1} \neq B_i$ incident to $u_{i+1}$ such that $w_{u_{i+1}}, \phi(B_{i+1}) > w_{u_{i+1}}, s(B_{i+1})$. Thus $B_{i+1} \neq B_i$, and also $u_{i+1} \neq u_i$, since $w_{B_i}, s(u_{i+1}) + w_{u_{i+1}}, \phi(B_i) \leq \mu(T)$ and $w_{B_i}, s(u_i) + w_{u_i}, \phi(B_i) > w_{B_i}, s(u_i) + w_{u_i}, s(B_i) = \mu(T)$. Therefore \{$B_0, u_1, B_1, u_2, B_2, \ldots$\} is an infinite path in the finite tree $\text{Inc}(T)$, which is impossible.

Thus there exists no homomorphism $\phi : T \to D^{\mu(T)+1}/\sim$, and so there exists a homomorphism $\psi : D^{\mu(T)+1}/\sim \to D$. Let $\delta : D \to D^{\mu(T)+1}$ be the diagonal map, and $q : D^{\mu(T)+1} \to D^{\mu(T)+1}/\sim$ the quotient map. By [5], Lemma 4.1, $D$ is rigid, whence the automorphism $\psi \circ q \circ \delta : D \to D$ is the identity. Therefore, $\psi \circ q : D^{\mu(T)+1} \to D$ is a near-unanimity polymorphism. \qed

**Lemma 4.3.** Let $A$ be a core structure with tree duality and $k \geq 2$ an integer. If the tree obstructions of $A$ all have degree of monstrosity of at most $k$, then $A$ admits a near-unanimity polymorphism of arity at most $k+1$.

**Proof.** We define the equivalence $\sim$ on $A^{k+1}$ as in the proof of Lemma 4.2. Thus there is a near-unanimity polymorphism of arity $k+1$ on $A$ if and only if $A^{k+1}/\sim \not\rightarrow A$.

Suppose that $A^{k+1}/\sim \not\rightarrow A$. Then there is a tree obstruction $T$ of $A$ such that $T \to A^{k+1}/\sim$. Since $\mu(T) \leq k$, the core $D$ of $D(T)$ admits a near-unanimity polymorphism of arity $k+1$ by Lemma 4.2; equivalently, $D^{k+1}/\sim \to D$. On one hand we get $T \not\rightarrow A$, thus $A \to D$ whence $A^{k+1}/\sim \to D^{k+1}/\sim \to D$. On the other hand, $T \to A^{k+1}/\sim$ implies $A^{k+1}/\sim \not\rightarrow D$, a contradiction. \qed

Combining the lower bound obtained in Lemma 3.1 with the upper bound obtained in Lemma 4.3, we get the following.

**Corollary 4.4.** Let $A$ be a core structure with finite duality, and $F = \{T_1, \ldots, T_m\}$ the set of its minimal tree obstructions. Then the smallest arity of a near-unanimity polymorphism on $A$ is $\max\{2, \mu(T_1), \ldots, \mu(T_m)\} + 1$.

The 2 in the above formula is necessary only when $\max\{\mu(T_1), \ldots, \mu(T_m)\} \leq 1$. The structures whose obstruction set satisfies the latter feature are not distinguished from the general structures with caterpillar duality from the point of view of near unanimity polymorphism. The next section provides the missing distinction, from the point of view of conservative polymorphisms.

**5. Medusa Duality and Conservative Polymorphisms**

In this section, we focus on trees with degree of monstrosity at most one, which we call *medusas*. A $\sigma$-structure $A$ is said to have *medusa duality* if its minimal obstructions are medusas. Note that for a $\sigma$-tree $T$, we have $\mu(T) = 0$ if and only if $T$ has a single hyperedge, say $(x_1, \ldots, x_r) \in R(T)$, where $R \in \sigma$. For a $\sigma$-structure $A$, we then have $T \not\rightarrow A$ if and only if $R(A) = \emptyset$. Thus a single-hyperedge
tree obstruction just forbids a relation that nonetheless appears in the type. The interesting medusas have degree of monstrosity of one; they have a single non-leaf “head” incident to all the “snake” hyperedges.

Given a structure $A$, an operation $\phi : A^n \to A$ is called conservative if $\phi(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for all $(a_1, \ldots, a_n) \in A^n$.

**Lemma 5.1.** Let $A$ be a core structure with medusa duality. Then all conservative operations on $A$ are polymorphisms.

**Proof.** Let $\phi : A^n \to A$ be a conservative operation on $A$. Define the structure $A_\phi$ on the universe $A$ as the smallest $\sigma$-structure such that $\phi : A^n \to A_\phi$ is a homomorphism. Formally, for every $R \in \sigma$ and $(X_1, \ldots, X_r) \in R(A^n)$, put $(\phi(X_1), \ldots, \phi(X_r)) \in R(A_\phi)$. Obviously we have $A \to A_\phi$; and since $A$ is rigid, the reverse $A_\phi \to A$ holds if and only if $\phi$ is a polymorphism on $A$.

Suppose that $A_\phi \not\subseteq A$. Then there is a medusa obstruction $T$ of $A$ such that there exists a homomorphism $\psi : T \to A_\phi$. Let $v$ be the head of $T$, and $\psi(v) = a \in A$. For every block $P = (R, (x_1, \ldots, x_r))$ of $T$, with $x_k = v$, we have $(\psi(x_1), \ldots, \psi(x_r)) \in R(A_\phi)$, so there exists $(X_1, \ldots, X_r) \in R(A^n)$ with $\phi(X_i) = \psi(x_i)$, $i = 1, \ldots, r$. Since $\phi$ is conservative, $\psi(x_i)$ is a coordinate of $X_i$ for every $i$, and in particular $a = \psi(v) = \psi(x_k)$ is a coordinate of $X_k$. By definition of $R(A^n)$, this means that $R(A)$ contains an hyperedge $(a_1, \ldots, a_r)$ with $a_k = a$. Thus $a$ is the head of a homomorphic image of $T$ in $A$, a contradiction.

Therefore, $A_\phi \to A$ whence $\phi$ is a polymorphism on $A$. \qed

The condition that all conservative operations are polymorphisms is much stronger than the existence of a near-unanimity polymorphism, since the near-unanimity identities requirements for polymorphisms can be extended in many ways to conservative operations. The converse of Lemma 5.1 essentially holds, apart from the fact that a structure may have relations with redundant coordinates. We will provide the correct version of the converse of Lemma 5.1 below, after we introduce the terminology needed to distinguish the subtle exceptions.

For a $\sigma$-structure $A$, $R \in \sigma$ of arity $r$ and $1 \leq i \leq r$, let

$$\pi_i(R(A)) = \{a \in A : \text{there exists } (a_1, \ldots, a_r) \in R(A) \text{ with } a_i = a\}$$

Note that $\pi_1(R(A)), \ldots, \pi_r(R(A))$ are either all empty (if $R(A) = \emptyset$) or all nonempty (otherwise). If $r \geq 2$, for $1 \leq i < j \leq r$ we define the binary relation $\beta_{R,i,j}(A)$ with domain $\pi_i(R(A))$ and image $\pi_j(R(A))$ by

$$\beta_{R,i,j}(A) = \{(a, b) : \text{there exists } (a_1, \ldots, a_r) \in R(A) \text{ with } a_i = a, a_j = b\}$$

**Lemma 5.2.** Let $A$ be a $\sigma$-structure which admits all conservative operations as polymorphisms. Then for $R \in \sigma$ of arity $r \geq 2$ and $1 \leq i < j \leq r$, either $\beta_{R,i,j}(A)$ is the total relation, or $|\pi_i(R(A))| \geq 2$, $\pi_j(R(A)) = \pi_i(R(A))$ and $\beta_{R,i,j}(A)$ is the equality relation on $\pi_i(R(A))$.

**Proof.** Suppose that $\beta_{R,i,j}(A)$ is neither a total relation nor an equality relation. Then there exist $a_i, b_j, c_j \in \pi_i(R(A))$ and $a_j, c_j \in \pi_j(R(A))$ such that $a_i \neq a_j$, $(a_i, a_j) \in \beta_{R,i,j}(A)$ and $(b_j, c_j) \not\in \beta_{R,i,j}(A)$. By definition of $\pi_i(R(A))$, $\pi_j(R(A))$, and $\beta_{R,i,j}(A)$, $R(A)$ contains three corresponding $r$-tuples $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_r)$, and $(c_1, \ldots, c_r)$. In particular $(a_i, b_i, c_i) \neq (a_j, b_j, c_j)$ whence there exists a conservative operation $\phi : A^n \to A$ such that $\phi(a_i, b_i, c_i) = b_i$, $\phi(a_j, b_j, c_j) = c_j$. Since
(φ(a_i, b_i, c_i), φ(a_j, b_j, c_j)) \not\in \beta_{R,i,j}(A)$, we then have $(φ(a_1, b_1, c_1), \ldots, φ(a_r, b_r, c_r)) \not\in R(A)$, thus $φ$ is not a polymorphism.

For a core $σ$-structure $A$ and $R \in σ$ of arity $r$, we can define an equivalence relation $∼_R$ on $\{1, \ldots, r\}$ by $i \sim j$ if $β_{R,i,j}(A)$ is an equality relation. If there are nontrivial $∼_R$-classes, they correspond to families of redundant, repeated coordinates in $R$. We can define a new relation $R′$ on $A$ by selecting one coordinate from each $∼_R$-class. We so define a new type $σ′$ and a $σ′$-structure $A′$ which has the same polymorphisms as $A$. In a sense, $A$ is just an artificially complicated version of $A′$.

**Theorem 5.3.** Let $A$ be a core $σ$-structure which admits all conservative operations as polymorphisms. Then $A$ has medusa duality, unless there exists $R \in σ$ of arity $r \geq 2$ and $1 \leq i < j \leq r$ such that $β_{R,i,j}(A)$ is not total. In the latter case, $A$ does not have finite duality.

**Proof.** Suppose that $β_{R,i,j}(A)$ is not total. Then there exist $a_i \in \pi_i(R(A)), a_j \in \pi_j(R(A))$ such that $a_i \neq a_j$ and $(a_i, a_j) \not\in β_{R,i,j}(A)$. For every $n \geq 1$, we define a $σ$-structure $B_n$ as follows. We take two copies $A’, A''$ of $A$; an element $a$ of $A$ is labeled $a'$ in $A'$ and $a''$ in $A''$. We then add to $R(B_n)$ $n$ hyperedges $H_k = (p^1_k, \ldots, p^n_k)$, $k = 1, \ldots, n$ and we identify $p^1_i$ to $a_i'$, $p^n_i$ to $a''_i$, and $p_k$ to $p_k^{r+1}$, $k = 1, \ldots, n−1$. Since $β_{R,i,j}(A)$ is the equality relation by Lemma 5.2, we have $B_n \not\rightarrow A$, but $B_n \nmid H_k \rightarrow A$ for $k = 1, \ldots, n$. Thus a $A$-obstruction mapping to $B_n$ must have at least $n$ hyperedges, whence $A$ does not have finite duality.

Now suppose that for all $R \in σ$ of arity $r \geq 2$ and $1 \leq i < j \leq r$, $β_{R,i,j}(A)$ is total. We will show that $A$ has finite duality using the following concept. The 1-tolerant power $A^n$ of $A$ is the $σ$-structure on the universe $A^n$ where for $R \in σ$, we put $(X_1, \ldots, X_r) \in R(A^n)$ if there are at least $n−1$ indices $i \in \{1, \ldots, n\}$ such that $(pr_i(X_1), \ldots, pr_i(X_r)) \in R(A)$. Following [5], Theorem 4.7, $A$ has finite duality if and only if for some integer $n$, there exists a homomorphism from $A^n$ to $A$.

Put $n = |A| + 1$. Let $φ : A^n \rightarrow A$ be any operation such that for every $X \in A^n$, $φ(X)$ appears at least twice as a coordinate of $X$. We show that $φ$ is a homomorphism from $A^n$ to $A$. Let $R \in σ$ and $(X_1, \ldots, X_r) \in R(A^n)$. By definition of $R(A^n)$, there exists a coordinate $ℓ$ such that for $k \neq ℓ$, $(pr_k(X_1), \ldots, pr_k(X_r)) \in R(A)$. For $i = 1, \ldots, r$, let $X'_i \in A^{n−1}$ be the vector obtained from $X_i$ by deleting its $ℓ$-th coordinate. Then $φ(X_i)$ is a coordinate of $X'_i$ for every $i$, and $(X'_1, \ldots, X'_r) \in R(A^{n−1})$. For every $i < j$ such that $X'_i = X'_j$ and $φ(X_i) ≠ φ(X_j)$, $β_{R,i,j}(A)$ is a total relation on a domain with at least two elements, so there exists $(a^{i,j}_1, \ldots, a^{i,j}_r) \in R(A)$ with $a^{i,j}_i \neq a^{i,j}_j$. Appending such elements $a^{j,i}_k$ as coordinates to $X'_k$, $k = 1, \ldots, r$, we extend $X'_1, \ldots, X'_r$ to vectors $X''_1, \ldots, X''_r$ such that $(X''_1, \ldots, X''_r) \in R(A^n)$ for some $m \geq n−1$, $X''_i = X''_j$ only if $φ(X_i) = φ(X_j)$, and $φ(X_i)$ is a coordinate of $X''_i$ for every $i$. Let $ψ : A^n \rightarrow A$ be a conservative operation such that $ψ(X''_i) = φ(X_i)$ for every $i$. Then $ψ$ is a polymorphism on $A$. Therefore $(φ(X_1), \ldots, φ(X_r)) = (ψ(X''_1), \ldots, ψ(X''_r)) \in R(A)$. This shows that $φ : A^n \rightarrow A$ is a homomorphism, therefore $A$ has finite duality.

Note that $A$ has near-unanimity polymorphisms $ν_k : A^k \rightarrow A$ for all $k \geq 3$, hence by Lemma 3.1 it has caterpillar duality. We prove that $A$ has in fact medusa duality. Suppose that $A$ has a minimal caterpillar obstruction $T$ which is not a medusa. Then $T$ contains a hyperedge $(t_1, \ldots, t_r) \in R(T)$ with two non-leaves $t_i, t_j$,
with $i < j$. Define $T_j$ as the maximal subtree of $T$ which contains $t_i$ as a non-leaf and $t_j$ as a leaf, and $T_j$ similarly by permuting indices. Then $T \neq T_i, T_j$, hence there exist homomorphisms $h_i : T_i \to A$ and $h_j : T_j \to A$. We then have $h_i(t_i) \in \pi_i(R(A))$, $h_j(t_j) \in \pi_j(R(A))$, and since $\beta_{R,i,j}(A)$ is total, there exists an hyperedge $(a_1, \ldots, a_r) \in R(A)$ such that $a_i = h_i(t_i)$ and $a_j = h_j(t_j)$. We can then define a homomorphism $h : T \to A$ by putting $h(t_k) = a_k$, $k = 1, \ldots, r$, $h(t) = h_i(t)$ for other elements $t$ of $T_i$, and $h(t) = h_j(t)$ for other elements $t$ of $T_j$. Thus $T \to A$, a contradiction. Therefore, all the minimal obstructions of $A$ are indeed medusas. \hfill \square

6. Concluding comments and problems

Following the arguments of [5], it can be shown that the structures with tree duality which admit near-unanimity polymorphisms also admit symmetric near-unanimity polymorphisms, though the existence of a symmetric near-unanimity polymorphism does not imply tree duality. However the main outstanding question is the validity of the converse of Lemma 4.3 for general core structures with tree duality. The partial converse Lemma 3.1 uses minimal obstructions in an essential way, and does not generalise readily. In particular, the following remains open.

For core structures with tree duality, is the existence of a near-unanimity polymorphism equivalent to the existence of a complete set of obstructions with bounded degree of monstrosity?

In [1], caterpillar duality is shown to be decidable, though it is not known whether the class of structures with this property coincides exactly with the class of structures with tree duality whose a core admits a majority polymorphism. Of course the property of admitting a majority polymorphism is also decidable. In contrast, neither the property of admitting a near-unanimity polymorphism nor the property of admitting a complete set of obstructions of bounded degree of monstrosity are known to be decidable for core structures with tree duality. And it is not known whether the two decision problems are connected.

References


\[9\] Near-unanimity polymorphisms
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