QUEEN’S UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 112
Final Examination
April, 2016
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Instructions: This is a 3-hour exam. There are 6 questions worth a total of 100 marks as indicated in the box below. Answer all questions in the space provided. If you need more room, answer on the back of the previous page. Show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise. Only CASIO FX-991 or Gold/Blue Sticker calculators are permitted.

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

<table>
<thead>
<tr>
<th>Question</th>
<th>Possible</th>
<th>Received</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Problem 2</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Problem 3</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Problem 4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Problem 5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Problem 6</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

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Problem 1. Multiple Choice (10 questions, 3 marks each)

Each question has four possible answers, labeled (A), (B), (C), and (D). Choose the most appropriate answer. Write your answer in the space provided. Write clearly using UPPERCASE letters. Illegible answers will be given a zero. You DO NOT need to justify your answer.

(1) The eigenvalues of \( A = \begin{bmatrix} 11 & 7 \\ 9 & 9 \end{bmatrix} \) are

(A) 11, 9  
(B) 19, 1  
(C) 18, 2  
(D) 4, 9

ANSWER TO (1): ______

(2) Let \( P \) and \( Q \) be matrices such that their product \( PQ \) is defined. If the size of \( PQ \) is 3 \( \times \) 7, then

(A) \( P \) must have 3 rows.  
(B) \( P \) must have 7 rows.  
(C) \( P \) must have 3 columns.  
(D) \( P \) must have 7 columns.

ANSWER TO (2): ______

(3) The area of the quadrilateral in \( \mathbb{R}^2 \) with vertices (0, 0), (1, 2), (3, 4), (4, 6) is

(A) 1  
(B) 2  
(C) 6  
(D) 12

ANSWER TO (3): ______
(4) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that first rotates points by 180° (counterclockwise) and then reflects points through the line $y = x$. Find the standard matrix for $T$.

(A) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

ANSWER TO (4): _______

(5) Let $A\vec{x} = \vec{b}$ be a linear system with $m$ equations and $n$ variables. Let $[A|\vec{b}]$ be its augmented matrix. Which one of the following is true?

(A) The system has a unique solution if rank$[A|\vec{b}]$ = rank$(A)$ = $m$.

(B) The system has a unique solution if rank$[A|\vec{b}]$ = rank$(A)$ = $n$.

(C) The system has a unique solution if $m = n$.

(D) The system has a unique solution if there is at least one free variable.

ANSWER TO (5): _______

(6) Let $A$ be a $4 \times 4$ matrix. If $\det(\text{adj} \, A) = 8$, find the value of $\det(A)$.

(A) 1

(B) 2

(C) 8

(D) 16

ANSWER TO (6): _______
(7) Determine which one of the following sets of vectors is linearly independent.

(A) \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}
\]

(B) \[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \quad
\begin{bmatrix}
3 \\
4 \\
5
\end{bmatrix}, \quad
\begin{bmatrix}
6 \\
7 \\
8
\end{bmatrix}
\]

(C) \[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}
\]

(D) \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
-1 \\
3
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}
\]

ANSWER TO (7): ____

(8) Which one of the following is not diagonalizable?

(A) \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]

(B) \[
\begin{bmatrix}
3 & 3 & -4 \\
0 & -4 & 2 \\
0 & 0 & -1
\end{bmatrix}
\]

(C) \[
\begin{bmatrix}
3 & 3 & -2 \\
3 & -4 & 0 \\
-2 & 0 & -1
\end{bmatrix}
\]

(D) \[
\begin{bmatrix}
2 & 4 & 3 \\
-4 & -6 & -3 \\
3 & 3 & 1
\end{bmatrix}
\]

ANSWER TO (8): ____
(9) Which one of the following is a subspace of $\mathbb{R}^2$?

(A) \( W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + y = 1 \right\} \)

(B) \( W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0 \right\} \)

(C) \( W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 = 0 \right\} \)

(D) \( W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 - y^2 = 0 \right\} \)

ANSWER TO (9): ______

(10) An \( n \times n \) matrix \( A \) is not invertible if

(A) the columns of \( A \) form a basis of \( \mathbb{R}^n \).

(B) the dimension of Null \( A \) is \( n \).

(C) the column vectors of \( A \) are linearly independent.

(D) the linear system \( A\vec{x} = \vec{b} \) is consistent for any \( \vec{b} \in \mathbb{R}^n \).

ANSWER TO (10): ______
Problem 2. Solve the linear system \( A\vec{x} = \vec{b} \) and write the solution in parametric vector form, where

\[
A = \begin{bmatrix}
2 & 1 & -1 & 6 \\
2 & 2 & -2 & 6 \\
4 & 0 & 0 & 13 \\
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}
6 \\
11 \\
3 \\
\end{bmatrix}
\]
Problem 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$
\begin{align*}
T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 2 \\ 5 \end{bmatrix}, & T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \\ -4 \\ 5 \end{bmatrix}, & T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 3 \\ 3 \\ 1 \\ 15 \end{bmatrix}
\end{align*}
$$

(1) Find the standard matrix of $T$.

(2) Find the pre-image of $\vec{0}$, i.e. find all $\vec{x} \in \mathbb{R}^3$ so that $T(\vec{x}) = \vec{0}$. Is $T$ one-to-one?
Problem 4.

Let $\vec{a}, \vec{b}$ be the columns of a $2 \times 2$ matrix $A$, i.e. $A = [\vec{a} \; \vec{b}]$. Suppose that

$$2\vec{a} - \vec{b} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \quad 5\vec{a} - 2\vec{b} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

Find $A$. 
Problem 5. Let $M_{2 \times 2}$ be the set of all the $2 \times 2$ matrices, it is a vector space with the usual addition of matrices and multiplication by scalars. Let $W$ be the set of all the $2 \times 2$ anti-symmetric matrices, i.e.

$$W = \{ A \in M_{2 \times 2} : A^T = -A \}$$

(1) Show that $W$ is a subspace of $M_{2 \times 2}$.

(2) Find a basis for $W$. 
Problem 6. Consider the difference equation

\[ x_{n+3} = -4x_n + 4x_{n+1} + x_{n+2}, \quad n = 0, 1, 2, 3, \ldots \]

with initial conditions \( x_0 = 0, x_1 = 1, x_2 = 7 \).

(1) Write the difference equation in a matrix form.

(2) Find the general formula for \( x_n \) in terms of \( n \).
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