Math 112, Homework #10

Question 1: (Bases for null space and column space) Find a basis for NulA and a basis for ColA for each of the following matrices. Check if the rank-nullity theorem is satisfied.

\[
A = \begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 2 & 0 \\
1 & 0 & 2 & 0 \\
0 & 1 & 2 & 0
\end{bmatrix}, \quad A = \begin{bmatrix}
1 & 0 & 2 & 4 \\
0 & 1 & -3 & -1 \\
3 & 4 & -6 & 8 \\
0 & -1 & 3 & 1
\end{bmatrix}
\]

Question 2: (Eigenvalues and eigenvectors) For each of the following matrices, find all the eigenvalues and the corresponding eigenvectors.

\[
A = \begin{bmatrix}
3 & -1 \\
-1 & 3
\end{bmatrix}, \quad A = \begin{bmatrix}
2 & 0 & -1 \\
4 & 1 & -4 \\
2 & 0 & -1
\end{bmatrix}, \quad A = \begin{bmatrix}
-1 & 4 & -2 \\
-3 & 4 & 0 \\
-3 & 1 & 3
\end{bmatrix}
\]

Question 3: Let \( A = \begin{bmatrix} 3 & k \\ -2 & 5 \end{bmatrix} \)

(1) Find conditions on \( k \) so that \( \lambda = -1 \) is an eigenvalue of \( A \).

(Hint: Use the fact that \( \lambda \) is an eigenvalue if and only if \( |A - \lambda I| = 0 \).)

(2) Find conditions on \( k \) so that all eigenvalues of \( A \) are real numbers.

(3) Find conditions on \( k \) so that \( A \) has exactly one (real) eigenvalue with multiplicity two. Find the eigenvalue.

(Hint: Find the characteristic equation for \( A \). It is a quadratic equation, determine when the quadratic equation has two distinct real roots, one repeated real root, or no real roots.)

Question 4: Let \( A = \begin{bmatrix} a & 1 & b \\ 1 & 0 & 0 \\ 0 & 1 & a \end{bmatrix} \),

1. Find conditions on \( a, b \) so that \( \lambda = -2 \) is an eigenvalue.

(Hint: Use the characteristic equation and the fact that \( \lambda \) is an eigenvalue if and only if \( |A - \lambda I| = 0 \).)
2. Find conditions on $a, b$ so that \[
\begin{bmatrix}
0 \\
3 \\
3
\end{bmatrix}
\] is an eigenvector and find the corresponding eigenvalue. (Hint: Use the definition of eigenvalues and eigenvectors: $A\vec{x} = \lambda \vec{x}$.)

Question 5:

1. Show that if $\lambda$ is an eigenvalue of $A$, then $\lambda^n$ is an eigenvalue of $A^n$ for any positive integer $n$.

2. Show that if $A$ is invertible and $\lambda$ is an eigenvalue of $A$, then $1/\lambda$ is an eigenvalue of $A^{-1}$.
   
   *Hint*: Use the definition of eigenvalues: $A\vec{x} = \lambda \vec{x}$.

3. Show that $A$ and $A^T$ have the same eigenvalues.
   
   *Hint*: Observe the characteristic equation of $A$ and $A^T$. 