Math 112, Homework #11

Part I: Please do the following questions from the textbook:

- §5.2: 27

Part II:

Question 1: Let $n = 0, 1, 2, \ldots$, consider the difference equation

$$
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} =
\begin{bmatrix}
0.8 & 0.3 \\
0.2 & 0.7
\end{bmatrix}
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix}
$$

with the initial conditions that

$$
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix} =
\begin{bmatrix}
1 \\
4
\end{bmatrix}.
$$

Find the general formula for $x_n$ and $y_n$.

Question 2: Let $n = 0, 1, 2, \ldots$, consider the difference equation

$$
\begin{bmatrix}
x_{n+1} \\
y_{n+1} \\
z_{n+1}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 3 \\
4 & 4 & 1
\end{bmatrix}
\begin{bmatrix}
x_n \\
y_n \\
z_n
\end{bmatrix}
$$

with the initial conditions that

$$
\begin{bmatrix}
x_0 \\
y_0 \\
z_0
\end{bmatrix} =
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix}.
$$

Find the general formula for $x_n$, $y_n$, $z_n$.

Question 3: Consider the difference equation $y_{n+3} = 3y_{n+2} + 4y_{n+1} - 12y_n$ with initial conditions $y_0 = 1, y_1 = 3, y_2 = 14$.

(1) Put the difference equation in matrix form.

(2) Find a general formula for $y_n$ in terms of $n$.

Question 4: Suppose that $A$ is a $2 \times 2$ matrix and

$$
A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

Find $A^7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hint: The conditions give information about the eigenvalues and eigenvectors of $A$. 
Question 5: (Similarity).

(a) If $A$ and $B$ are similar matrices of size $n \times n$, show that they have the same determinant.

(b) Are the two matrices $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$ similar?

(c) Suppose $P, Q$ are square matrices of the same size and $P$ is invertible. Let $A = PQ$ and $B = QP$, show that $A$ and $B$ are similar.

**Hint:** Compare $A$ and $PBP^{-1}$

(d) If $A$ and $B$ are similar, and $B$ and $C$ are similar, are $A$ and $C$ similar?