Determinant
Math 112, week 6

Goals:

• Invertible matrix and determinant.
• Determinant and elementary row operations.
• Transpose and Multiplicative property of determinant.

Suggested Textbook Readings: Sections §3.1, §3.2
**Invertible 2 \times 2 matrix.**

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then \( A \) is invertible \( \iff \) \( ad - bc \neq 0 \).

**Invertible 3 \times 3 matrix.**

Let \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \), then \( A \) is invertible \( \iff \).
Notation: $A_{ij}$ is the submatrix obtained from $A$ by deleting the $i$-th row and $j$-th column of $A$.

**Determinant is defined recursively:**

- For $n = 1$, the determinant of a $1 \times 1$ matrix $A = [a_{11}]$ is 
  \[
  \det A = a_{11}
  \]

- For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]_{n \times n}$ is 
  \[
  \det A = \]

**Theorem:** The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down by any column:

- $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$ (expansion across row $i$)
- $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ (expansion down column $j$)

$C_{ij} = (-1)^{i+j} \det A_{ij}$

Use a matrix of signs to determine $(-1)^{i+j}$:

\[
\begin{pmatrix}
+ & - & + & \cdots \\
- & + & - & \cdots \\
+ & - & + & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]
Example 1: Let $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix}$, find $\det A$ by

- **Expansion across row-1:**
- **Expansion across row-3:**
- **Expansion down column-3:**

Example 2: Compute $\det A$ where

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 3 & 0 & -1 \\ -2 & -2 & 0 & 3 \\ 0 & 5 & 0 & -4 \end{bmatrix}$$
Triangular Matrices:

**Theorem:** If $A$ is a triangular matrix, then $\det A$ is the product of the main diagonal entries of $A$.

**Example 3:** Compute

\[
\begin{vmatrix}
2 & 3 & 4 & 5 & 6 \\
0 & 1 & 3 & 0 & 2 \\
0 & 0 & -2 & 4 & 0 \\
0 & 0 & 0 & 3 & 9 \\
0 & 0 & 0 & 0 & 5
\end{vmatrix}
\]
Determinant and elementary row operation.

**Fact:** Let $A$ be a square matrix, $A \xrightarrow{E.R.O} B$

1. a multiple of one row is added to another row:

2. two rows are interchanged:

3. one row is multiplied by $k$:

**Example 4:** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $\det B$.

1. $B = \begin{bmatrix} a & b \\ 4a + c & 4b + d \end{bmatrix}$

2. $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

3. $B = \begin{bmatrix} a & b \\ -2c & -2d \end{bmatrix}$

Fact 3 indicates that

\[
\begin{vmatrix} k a & k b & k c \\ a & b & c \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ * & * & * \\ * & * & * \end{vmatrix}
\]
Example 5: Compute

\[
\begin{vmatrix}
1 & 2 & 3 & 3 \\
2 & 3 & 1 & 1 \\
4 & 7 & 6 & 8 \\
7 & 9 & 2 & 4 \\
\end{vmatrix}
\]

Example 6: Compute

\[
\begin{vmatrix}
0 & 1 & 2 & -1 \\
2 & 5 & -7 & 3 \\
0 & 3 & 6 & 2 \\
-2 & -5 & 4 & 2 \\
\end{vmatrix}
\]
Suppose $A$ has been row reduced to REF $U$ by row replacements and row interchanges.

$$
U = \begin{bmatrix}
\begin{array}{cccc}
\blacksquare & \ast & \ast & \ast \\
0 & \blacksquare & \ast & \ast \\
0 & 0 & \blacksquare & \ast \\
0 & 0 & 0 & \blacksquare \\
\end{array}
\end{bmatrix}
$$

$$
det A = \begin{cases} 
\text{when } A \text{ is not invertible} \\
\text{when } A \text{ is invertible}
\end{cases}
$$

**Theorem:** A square matrix $A$ is invertible $\iff$ $\det A \neq 0$.

A square matrix is called:

- **singular:**

- **non-singular:**
Properties of determinant.

**Theorem:** $\det A^T = \det A$.

**Partial proof of** $\det A = \det A^T$:

\[
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix}
= \\

\begin{vmatrix}
a & d & g \\
b & e & h \\
c & f & i \\
\end{vmatrix}
= \\

Determinant and column operations:

Let $A$ be a square matrix, $A \rightarrow B$

1. a multiple of one *column* is added to another *column*:

2. two *columns* are interchanged:

3. one *column* is multiplied by $k$: 
Theorem: (Multiplicative Property): If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = (\det A)(\det B)$.

Example 7: Compute $\det(A^3)$ if $\det A = 4$.

Example 8: Suppose $\det A = 2$, find $\det A^{-1}$.

Example 9: For $n \times n$ matrices $A$ and $B$, suppose $\det A = 2$ and $\det B = 3$, find $\det(A^T B^{-1} A)$.

Example 10: Let $A$ be an $n \times n$ matrix, if $\det A = a$, find $\det(rA)$, where $r$ is a scalar.
**Theorem:** If $A$ is an $n \times n$ matrix, $E$ is an $n \times n$ elementary matrix, then

$$|EA| = |E||A|$$

$$|E| = \begin{cases} 
\text{if } E \text{ is add to a row a multiple of another row} \\
\text{if } E \text{ is interchange of two rows} \\
\text{if } E \text{ is multiply a row by } k 
\end{cases}$$

**Proof of** $|AB| = |A||B|$: 

**Case 1:** $A$ is not invertible.

**Case 2:** $A$ is invertible.