Some applications of determinant
Math 112, week 7

Goals:

• Cramer’s rule.
• A formula for the inverse of a matrix.
• Determinant as area or volume.

Suggested Textbook Readings: Sections §3.3
Cramer’s rule
Let $A$ be an invertible $n \times n$ matrix. Then the unique solution of $A\vec{x} = \vec{b}$ is given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}, \quad i = 1, 2, \ldots, n$$

where $A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{b} & \cdots & \vec{a}_n \end{bmatrix}$

Example 1: Solve the system by Cramer’s rule.

$$\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + 5x_3 = 0 \\ x_2 - x_3 = 4 \end{cases}$$
Example 2: Suppose $s$ is an unspecified parameter. Determine the values of $s$ for which the system has a unique solution, and use Cramer’s rule to describe the solution.

\[
\begin{align*}
3sx_1 - 2x_2 &= 4 \\
-6x_1 + sx_2 &= 1
\end{align*}
\]

Proof of Cramer’s rule:
A formula for $A^{-1}$: Let $A$ be an invertible matrix, then

$$A^{-1} = \frac{1}{\det A} \text{adj} A$$

adjoint of $A$: \[
\text{adj} A = \begin{bmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{bmatrix}
\]

Proof:
Example 3: *Use the formula to find the inverse of*

\[
A = \begin{bmatrix}
2 & 1 & 3 \\
1 & -1 & 1 \\
1 & 4 & -2
\end{bmatrix}
\]
Example 4: A linear system is given below.

\[
\begin{align*}
x_1 + 2x_2 + x_3 &= 1 \\
2x_2 - x_3 &= -2 \\
-x_1 - x_2 + x_3 &= -1
\end{align*}
\]

(a) What is the determinant of the coefficient matrix \( A \)?

(b) Find the cofactor matrix of \( A \).

(c) Find the inverse of \( A \).

(d) Use the inverse matrix found in (c) to solve the system.
Determinant as area or volume:

- If \( A \) is a \( 2 \times 2 \) matrix, the area of the parallelogram determined by the columns of \( A \) is \(| \det A |\).
- If \( A \) is a \( 3 \times 3 \) matrix, the volume of the parallelepiped determined by the columns of \( A \) is \(| \det A |\).

Example 5: (1) Find the area of the parallelogram whose vertices are \((0, 0), (5, 2), (6, 4), (11, 6)\).

(2) Find the volume of the parallelepiped with vertices \((0, 0, 0), (1, 0, -3), (1, 2, 4), (5, 1, 0)\)

(3) Find a formula for the area of the triangle whose vertices are the origin, \( \vec{v}_1, \vec{v}_2 \) in \( \mathbb{R}^2 \).
**Linear transformation and area, volume.** Suppose $S$ is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A$, then
  \[ \text{area of } T(S) = |\det A| \cdot \text{area of } S \]

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with standard matrix $A$, then
  \[ \text{volume of } T(S) = |\det A| \cdot \text{volume of } S \]

**Example 6:** Find the area of the region $E$ bounded by ellipse whose equation is
\[
\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1
\]