1. Consider
\[ x' = Ax, \quad A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \]

(a) Use eigensolutions to find a fundamental matrix of the differential equation.
(b) Find \( e^{At} \).

Solution.

(a) The characteristic equation is:
\[ \det(A - rI) = -(r - 3)^2(r + 1) \]
So the three eigenvalues are
\[ r_1 = -1, \quad r_2 = r_3 = 3 \]

For \( r_1 = -1 \), an eigenvector \( v_1 \) satisfies:
\[ (A - r_1 I)v_1 = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]
Choose \( v_1 = \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix} \).

For \( r_2 = r_3 = 3 \), its eigenvector satisfies:
\[ (A - r_2 I)v = \begin{pmatrix} -2 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]
we can find two linearly independent eigenvectors:
\[ v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix} \]
We have found three solutions:

\[ x_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix}, \quad x_2(t) = e^{3t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_3(t) = e^{3t} \begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix} \]

Their Wronskian

\[ W(x^{(1)}, x^{(2)}, x^{(3)}) = \det(x^{(1)}, x^{(2)}, x^{(3)}) = e^{5t} \neq 0 \]

So they form a fundamental set of solutions, and a fundamental matrix is:

\[
M(t) = \begin{pmatrix}
    e^{-t} & 0 & e^{3t} \\
    0 & e^{3t} & 0 \\
    -e^{-t}/2 & 0 & e^{3t}/2
\end{pmatrix}
\]

(b) We can find the matrix exponential use the following equality:

\[ e^{At} = M(t)M(0)^{-1} \]

Since

\[
M(0) = \begin{pmatrix}
    1 & 0 & 1 \\
    0 & 1 & 0 \\
    -1/2 & 0 & 1/2
\end{pmatrix} \Rightarrow M(0)^{-1} = \begin{pmatrix}
    1/2 & 0 & -1 \\
    0 & 1 & 0 \\
    1/2 & 0 & 1
\end{pmatrix}
\]

We get

\[
e^{At} = M(t)M(0)^{-1} = \begin{pmatrix}
    (1/2)e^{-t} + (1/2)e^{3t} & 0 & -e^{-t} + e^{3t} \\
    0 & e^{3t} & 0 \\
    -(1/4)e^{-t} + (1/4)e^{3t} & 0 & (1/2)e^{-t} + (1/2)e^{3t}
\end{pmatrix}
\]

2. Solve the initial value problem:

\[ x' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \]

**Solution.** We solve this equation by finding \( e^{At} \),

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = I + N
\]
where $I$ is the $3 \times 3$ identity matrix and $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$.

Since $IN = NI$, we have $e^{At} = e^{(I+N)t} = e^{It}e^{Nt}$ and

$$e^{Nt} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} = e^tI$$

Computing the powers of $N$, we get

$$N^2 = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N^3 = 0, \quad N^4 = 0, \ldots$$

so

$$e^{Nt} = I + tN + \frac{t^2}{2!}N^2 + 0 + \cdots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2t & 3t \\ 0 & 0 & 4t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 4t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2t & 3t + 4t^2 \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{pmatrix}$$

So the matrix exponential

$$e^{At} = e^{It}e^{Nt} = (e^tI)e^{Nt} = e^t \begin{pmatrix} 1 & 2t & 3t + 4t^2 \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{pmatrix}$$

The solution that satisfies the initial condition is

$$x(t) = e^{At} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} 3 + 7t + 4t^2 \\ 2 + 4t \\ 1 \end{pmatrix}$$

3. If the matrix $A$ is block diagonal, i.e. $A$ has the following form

$$A = \begin{pmatrix} A_1 \\ & A_2 \\ & & \ldots \\ & & & A_n \end{pmatrix}$$
where each $A_i$ is a square matrix of any size. Then

$$e^{At} = \begin{pmatrix} e^{A_1t} & 0 & \cdots & 0 \\ 0 & e^{A_2t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{A_nt} \end{pmatrix}$$

Use this property to find $e^{At}$, where

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution.

$$e^{At} = \begin{pmatrix} e^{3t} \cos t & -e^{3t} \sin t & 0 & 0 \\ e^{3t} \sin t & e^{3t} \cos t & 0 & 0 \\ 0 & 0 & e^t & 2te^t \\ 0 & 0 & 0 & e^t \end{pmatrix}$$

4. Consider the linear system

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a real constant matrix. Recall that

$$\text{tr}A = a + d, \quad \det A = ad - bc$$

Define $\Delta = (\text{tr}A)^2 - 4\det A$. Show that:

(a) If $r_1$ and $r_2$ are the eigenvalues of $A$, then

$$\text{tr}A = r_1 + r_2, \quad \det A = r_1r_2$$

(b) If $\det A > 0$ and $\Delta \geq 0$, then the critical point $(0,0)$ is a node.

(c) If $\det A < 0$, then the critical point $(0,0)$ is a saddle.

(d) If $\text{tr}A \neq 0$ and $\Delta < 0$, then the critical point $(0,0)$ is a spiral point.
(e) If $trA = 0$ and $det A > 0$, then the critical point $(0, 0)$ is a center.

(f) If $trA < 0$ and $det A > 0$, then the critical point $(0, 0)$ is asymptotically stable.

(g) If $trA > 0 \ or \ det A < 0$, then the critical point $(0, 0)$ is unstable.

Solution.

(a) The characteristic equation of $A$ is
\[
det(A - rI) = (a - r)(d - r) - bc = r^2 - (a + d)r + (ad - bc) = 0
\]
which is:
\[
r^2 - (trA)r + det A = 0
\]
If $r_1, r_2$ are the eigenvalues of $A$, then
\[
r^2 - (trA)r + det A = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1r_2
\]
So $trA = r_1 + r_2, \ det A = r_1r_2$.

(b) When $det A > 0$ and $\Delta \geq 0$, $A$ has two real eigenvalues with the same sign. So the origin may be a nodal sink, nodal source, improper node or proper node (i.e. star point). In general, it is a node.

(c) When $det A < 0$, $A$ has two real eigenvalues with opposite signs, so the origin is a saddle.

(d) When $trA \neq 0$ and $\Delta < 0$, $A$ has two complex conjugate eigenvalues with non-zero real part, so the origin is a spiral point.

(e) When $trA = 0$ and $det A > 0$, $A$ has two pure imaginary (i.e. complex with zero real part) eigenvalues, so the origin is a center.

(f) When $trA < 0$ and $det A > 0$, $A$ has either two negative eigenvalues, or it has complex conjugate eigenvalues with negative real parts. In either case, the origin is asymptotically stable.

(g) When $trA > 0$, if $A$ has real eigenvalues, then at least one of them is positive; if $A$ has complex conjugate eigenvalues, then the real part is positive. In either case, the origin is unstable. When $det A < 0$, we know the origin is a saddle, so it is unstable.