Part I

Section 2.1, Problem 30

Solution. This is a first-order linear equation in standard form. Computing the integrating factor, we get

\[ \mu(t) = e^{-t} \]

Multiply \( \mu(t) \) to both sides of the equation, we get

\[ e^{-t}y - e^{-t}y = e^{-t}(1 + 3 \sin t) \]

\[ \Rightarrow [e^{-t}y]' = e^{-t}(1 + 3 \sin t) \]

Thus

\[ e^{-t}y = \int e^{-t}(1 + 3 \sin t) dt + C = -e^{-t} + 3 \int e^{-t} \sin t dt + C \]

Computing

\[ \int e^{-t} \sin t dt = - \int \sin t d(e^{-t}) \]

\[ = -e^{-t} \sin t + \int e^{-t} \cos t dt \]

\[ = -e^{-t} \sin t - \int \cos t d(e^{-t}) \]

\[ = -e^{-t} \sin t - [e^{-t} \cos t + \int e^{-t} \sin t dt] \]

\[ = -e^{-t} \sin t - e^{-t} \cos t - \int e^{-t} \sin t dt \]

Hence \( \int e^{-t} \sin t dt = -(1/2)e^{-t}(\sin t + \cos t) \), and the general solution is:

\[ y(t) = -1 - (3/2)(\sin t + \cos t) + Ce^t \]

The initial condition is \( y(0) = y_0 \), so we get \( C = y_0 + 5/2 \), the solution to the initial value problem is

\[ y(t) = -1 - (3/2)(\sin t + \cos t) + (y_0 + 5/2)e^t \]

Since \( -1 - (3/2)(\sin t + \cos t) \) will remain finite for any \( t \), and \( e^t \to \infty \) as \( t \to \infty \), we know the solution will remain finite when \( t \to \infty \) only if \( (y_0 + 5/2) = 0 \), i.e. \( y_0 = -5/2 \).
Section 2.1, Problem 37

Solution. Consider a first order linear ODE with constant coefficient:

\[ x' + ax = g(t) \]

Where \( a \) is any positive constant. Without loss of generality, let \( a = 1 \). The general solution of

\[ x' + x = g(t) \]

is

\[ x(t) = e^{-t}[\int g(t)e^t \, dt + C] = e^{-t} \int g(t)e^t \, dt + Ce^{-t} \]

Since \( Ce^{-t} \to 0 \) as \( t \to \infty \), it is enough to choose \( g(t) \) so that

\[ e^{-t} \int g(t)e^t \, dt = 4 - t^2 \]

then all the solutions of the equation will approach the curve \( x = 4 - t^2 \) as \( t \to +\infty \).

To find \( g(t) \) so that

\[ \int g(t)e^t \, dt = e^t(4 - t^2) \]

we take derivatives on both sides: \( g(t)e^t = e^t(4 - t^2 - 2t) \), so \( g(t) = 4 - t^2 - 2t \).

The differential equation we construct is then

\[ x' + x = 4 - t^2 - 2t \]

Its general solution is \( x(t) = (4 - t^2) + Ce^{-t} \).

When \( t \to \infty \), \( x(t) \) approaches the curve \( x = 4 - t^2 \) as required.
Section 2.3, Problem 16

Solution. Let $T(t)$ be the temperature of the coffee at time $t$, then it satisfies the equation:

$$T' = -k(T - 70), \quad T(0) = 200$$

where $k$ is the conduction constant to be determined. The solution of the initial value problem is

$$T(t) = 70 + 130e^{-kt}$$

We know when $t = 1$ min, the temperature of the coffee is 190, hence $T(1) = 190$; use this condition to find the value of $k$:

$$T(1) = 70 + 130e^{-k} = 190 \quad \Rightarrow \quad k = \ln(13/12)$$

So the temperature of the coffee at time $t$ is given by

$$T(t) = 70 + 130e^{-\ln(13/12)t}$$

Let $T(t) = 150$, we get $t = \ln(13/8)/\ln(13/12) \approx 6.07$ min.

Section 2.5, Problem 26

Solution.

(a) Equilibrium solutions are given by the values of $x$ such that $f(x) = x(a - x^2) = 0$.

- When $a \leq 0$, equilibrium solution is 0.
- When $a > 0$, equilibrium solution is 0, $\pm \sqrt{a}$.

Phase lines:

- When $a < 0$: $f(x) > 0$ when $x < 0$ and $f(x) < 0$ when $x > 0$, so its phase line is:

  \[ \begin{array}{c}
  \cdots \rightarrow \bullet \leftarrow \rightarrow \cdots \\
  0 \quad x
  \end{array} \]

- When $a = 0$: $0, f(x) > 0$ when $x < 0$ and $f(x) < 0$ when $x > 0$, so its phase line is:

  \[ \begin{array}{c}
  \cdots \rightarrow \bullet \leftarrow \rightarrow \cdots \\
  0 \quad x
  \end{array} \]

- When $a > 0$, the signs of $f(x)$ between the equilibrium points are:

  \[ \begin{array}{c}
  + \quad - \quad + \quad - \\
  \cdots \bullet \leftarrow \bullet \rightarrow \cdots \\
  -\sqrt{a} \quad 0 \quad \sqrt{a} \quad x
  \end{array} \]
So the phase line is:

\[ -\sqrt{a} \quad 0 \quad \sqrt{a} \quad x \]

Stability properties:
- When \( a < 0 \), 0 is asymptotically stable.
- When \( a = 0 \), 0 is asymptotically stable.
- When \( a > 0 \), 0 is unstable; \( \pm \sqrt{a} \) are asymptotically stable.

(c) See the following bifurcation diagram.

Note: The solid blue curve and the dashed orange line is the location of the equilibrium points versus the parameter \( a \). The solid black lines with arrows are the phase lines corresponding to different range of \( a \). In a bifurcation diagram, we often plot the stable equilibrium points in solid curves, while the unstable equilibrium points in dashed curves.

The bifurcation diagram of \( x' = x(a - x^2) \) resembles a pitch fork, so we call it the pitchfork bifurcation.

Part II

1. A tank initially contains 60 gal of pure water. A mixture containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 hour.

   (a) Find the amount of salt in the tank after \( t \) minutes.

   (b) What is the maximum amount of salt ever in the tank?

Solution.

(a) Let \( x(t) \) be the amount of salt in the tank at time \( t \), and we only consider time \( t < 60 \), then it satisfies the equation:

\[
x' = 2 - \frac{3x}{60 - t}, \quad x(0) = 0, \quad t < 60
\]
The equation in standard form is

$$x' + \frac{3x}{60 - t} = 2$$

the integrating factor is

$$\mu(t) = e^{\int \frac{3}{60 - t}dt} = e^{-3\ln|60 - t|} = (60 - t)^{-3}$$

Multiply the integrating factor to the equation we get:

$$[(60 - t)^{-3}x]' = 2(60 - t)^{-3}$$

Since

$$\int 2(60 - t)^{-3}dt = (60 - t)^{-2}$$

The general solution is:

$$x(t) = \frac{(60 - t)^{-2} + C}{(60 - t)^{-3}} = (60 - t) + C(60 - t)^3$$

Substitute the initial condition $x(0) = 0$ we get $C = -1/60^2$. The amount of salt in the tank after $t$ minutes is

$$x(t) = (60 - t) - (60 - t)^3/3600$$

(b) To find the maximum value of $x(t)$, we locate its critical points first:

$$x'(t) = -1 + \frac{1}{1200}(60 - t)^2 = 0$$

The critical point is $t = 60 - 10\sqrt{12}$, and $x''(60 - 10\sqrt{12}) < 0$, so $x(t)$ achieves its maximum at $t = 60 - 10\sqrt{12}$; the maximum amount of salt ever in the tank is

$$x(60 - 10\sqrt{12}) = \frac{20}{3}\sqrt{12} \approx 23.09 \text{ lb}$$